Math 201, Spring 2022

Problem Set # 6

Due March 16, 2022 at 11:59pm on gradescope

Question 1. Let $Z \sim \mathcal{N}(0,1)$ and $X \sim \mathcal{N}(\mu, \sigma^2)$. This means that Z is a standard normal random variable with mean 0 and variance 1, while X is a normal random variable with mean μ and variance σ^2.

a) Calculate $E[Z^3]$ using integration by parts (as in the proof of Fact 3.59 in the book).

b) Calculate $E[X^3]$. **Hint:** Do not integrate with the density function of X unless you like messy integration. Instead use the fact that $X = \sigma Z + \mu$ and expand the cube inside the expectation.

a) We have

$$E[Z^3] = \int_{-\infty}^{\infty} x^3 f(x)dx \qquad = \int_{-\infty}^{\infty} x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

This is an improper integral and we should interpret it as

$$E[Z^3] = \lim_{L\to\infty} \int_{-L}^{L} x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Notice that the integrand is an odd function since

$$(-x)^3 f(-x) = -x^3 f(x)$$

thus for finite L we have

$$\int_{-L}^{L} x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0.$$

To pass to the limit $L \to \infty$ we must make sure that the integral converges absolutely, i.e.,

$$\int_{-\infty}^{\infty} \left| x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| dx < \infty.$$

Computing this integral we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left| x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| dx &= \lim_{L\to\infty} \int_{-L}^{L} \left| x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| dx \\
&= \lim_{L\to\infty} \int_{-L}^{L} |x|^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \lim_{L\to\infty} 2 \int_{0}^{L} x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \frac{2}{\sqrt{2\pi}} \lim_{L\to\infty} \int_{0}^{L} x^3 e^{-x^2/2} dx
\end{aligned}$$

where between lines two and three we used the fact that the integrand is even. Now we use integration by parts with $u = x^2, du = 2x$ and $dv = xe^{-x^2/2}, v = -e^{-x^2/2}$ to compute the integral

$$\begin{aligned}\int_0^L x^3 e^{-x^2/2} dx &= -x^2 e^{-x^2/2}\Big|_0^L + 2\int_0^L xe^{-x^2/2} dx \\ &= -L^2 e^{-L^2/2} - 2\int_0^L \frac{d}{dx} e^{-x^2/2} dx \\ &= -L^2 e^{-L^2/2} - 2e^{-L^2/2} + 2.\end{aligned}$$

Now plugging these terms back in to the original integral we get

$$\begin{aligned}\int_{-\infty}^{\infty} \left| x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| dx &= \frac{2}{\sqrt{2\pi}} \lim_{L\to\infty} \left(-L^2 e^{-L^2/2} - 2e^{-L^2/2} + 2 \right) \\ &= \frac{4}{\sqrt{2\pi}}.\end{aligned}$$

Since this is finite we conclude that the integral converges absolutely and therefore

$$E[Z^3] = 0.$$

b) Notice that

$$Z = \frac{X - \mu}{\sigma}$$

is a standard normal random variable and solving for X we have

$$X = \sigma Z + \mu.$$

Then we have

$$\begin{aligned}E[X^3] &= E[(\sigma Z + \mu)^3] \\ &= E[\sigma^3 Z^3 + 3\sigma^2 Z^2 \mu + 3\sigma Z \mu^2 + \mu^3] \\ &= \sigma^3 E[Z^3] + 3\sigma^2 \mu E[Z^2] + 3\sigma\mu^2 E[Z] + \mu^3 \\ &= 3\sigma^2 \mu + \mu^3.\end{aligned}$$

Question 2. My bus is scheduled to depart at noon. However, we're in Randomland, and this means the departure time varies randomly with average departure time 12 o'clock and a standard deviation of 6 minutes. Assume the departure time is normally distributed. If I get to the bus stop 5 minutes past noon, what is the probability that the bus has not yet departed?

First let's shift change coordinates so that a time of 0 means 12 o'clock and measure time in minutes. Let X be the time at which the bus departs, $X \sim N(0, 6^2)$. We want to find $P(X > 5)$. The easiest way to solve this is to notice that if we define $Z = \frac{X}{6}$ then $Z \sim N(0, 1)$. Now write the probability of interest in terms of Z

$$
\begin{aligned}
P(X > 5) &= P\left(X/6 > 5/6\right) \\
&= P(Z > 5/6) \\
&= 1 - P(Z \leq 5/6) \\
&= 1 - \Phi(5/6) \\
&\approx 1 - 0.7967 = 0.2033
\end{aligned}
$$

where $\Phi(s)$ is the CDF of the standard normal distribution and we used the table in Appendix E of the textbook to compute $\Phi(.83) \approx 0.7967$.