Math 201, Spring 2022

Problem Set # 5

Due March 2, 2022 at 11:59pm on gradescope

Question 1. a) Let X be a nonnegative integer-valued random variable, i.e., it takes values in $\{0, 1, 2, ...\}$. Show that $E[X] = \sum_{k=1}^{\infty} P(X \geq k)$

b) Assume now that $X \sim \text{Geom}(1/3)$, use the formula from (a) to calculate $E[X]$.

Solution.

a) For a discrete random variable we have

$$\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} kP(X = k) \\
&= \sum_{k=1}^{\infty} kP(X = k) \\
&= P(X = 1) + P(X = 2) + P(X = 2) + P(X = 3) + P(X = 3) + P(X = 3) + \cdots \\
&= P(X = 1) + P(X = 2) + P(X = 3) + \cdots + P(X = 2) + P(X = 3) + \cdots \\
&= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \cdots \\
&= \sum_{k=1}^{\infty} P(X \geq k).
\end{aligned}$$

Now to make this computation more formal we can compute as follows:

$$\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} kP(X = k) \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{k} P(X = j) \\
&= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} P(X = j) \\
&= \sum_{j=1}^{\infty} P(X \geq j).
\end{aligned}$$

b) For a geometric random variable with parameter p we have

$$P(X \geq k) = (1 - p)^{k-1}$$

and using the formula from part (a),

$$\begin{aligned} E[X] &= \sum_{k=1}^{\infty} P(X \geq k) \\ &= \sum_{k=1}^{\infty} (1-p)^{k-1} \\ &= \frac{1}{1-(1-p)} \\ &= \frac{1}{p}. \end{aligned}$$

Question 2. Buses arrive at ten minute intervals starting at noon. A man arrives at the bus stop a random number X minutes after noon, where X has distribution function

$$P(X \leq x) = \begin{cases} 0 & \text{if } x < 0, \\ x/60 & \text{if } 0 \leq x \leq 60 \\ 1 & \text{if } x > 60. \end{cases}$$

What is the probability that he waits less than five minutes for a bus?

Solution. Let Y be the amount of time he waits and let X be his arrival time at the bus stop. We are trying to find $P(Y \leq 5)$. We break up the hour into 10 minute chunks and we have

$$Y = \begin{cases} 10 - X & \text{if } 0 < X \leq 10 \\ 20 - X & \text{if } 10 < X \leq 20 \\ 30 - X & \text{if } 20 < X \leq 30 \\ \vdots & \\ 60 - X & \text{if } 50 < X \leq 60. \end{cases}$$

Then the probability of interest is

$$\begin{aligned} P(Y \leq 5) &= P(10 - X \leq 5 \cap 0 < X \leq 10) + P(20 - X \leq 5 \cap 10 < X \leq 20) + \cdots + P(60 - X \leq 5 \cap 50 < X \leq \\ &= P(5 \leq X \leq 10) + P(15 \leq X \leq 20) + \cdots + P(55 \leq X \leq 60) \\ &= 6P(5 \leq X \leq 10) = 6 \cdot \frac{5}{60} = \frac{1}{2}. \end{aligned}$$

Where between lines 2 and 3 we used the fact that X has the uniform distribution on the interval $[0, 60]$.