Math 201, Spring 2022
Problem Set # 2

Question 1. Suppose Ram and Laxman play a game: they take turns shooting arrows at a bullseye. Ram goes first, and if he misses, Laxman goes next. If Laxman misses as well, the round ends, and the next round begins where Ram again goes first. Ram hits the bulls eye with probability 1/3, and Laxman hits with probability 1/2. The game ends when one of them hits the bullseye. Let X be the total number of rounds played (a round is one in which either Ram hits and wins, Ram misses and Laxman hits and wins, or both players miss).

(a) Find the chance that Ram wins on the k-th round.

Solution. The chance that neither player wins on a given round is $(2/3)(1/2) = 1/3$. So the chance that we get to the k-th round is $(1/3)^{k-1}$. The chance that Ram then wins on the k-th round is $1/3$. So we get $(1/3)^k$.

(b) Find the probability that Ram wins.

Solution. The chance that Ram wins in the k-th round is $(1/3)^k$ (as above, $k-1$ rounds where no one wins followed by Ram winning). So we get

$$\sum_{k=1}^{\infty}(1/3)^{k-1}(1/3) = \frac{1}{3}\sum_{n=0}^{\infty}(1/3)^n = \frac{1/3}{1-(1/3)} = \frac{1}{2}.$$

Question 2. Imagine a game of 3 players where exactly one player wins in the end and all players have equal chances of being the winner. The game is repeated four times. Find the probability that there is at least one person who wins no games. **Hint:** Consider the events A_i in which person i wins no games and use the inclusion-exclusion formula.

Solution. Following the hint let A_i be the event in which person i wins no games. We are interested in computing

$$P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} A_i - \sum_{1 \le i,j \le 3} P(A_i \cap A_j) - P(A_1 \cap A_2 \cap A_3)$$

where we have used the inclusion-exclusion formula. Now we compute each term in the sum.

$$P(A_i) = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$
$$P(A_i \cap A_j) = \frac{1}{3^4} = \frac{1}{81}$$
$$P(A_1 \cap A_2 \cap A_3) = 0.$$

Summing up the terms we find

$$P(A_1 \cup A_2 \cup A_3) = 3 \cdot \frac{16}{81} - 3 \cdot \frac{1}{81} = \frac{5}{9}.$$