Math 201, Spring 2022

Problem Set # 12

Due April 26, 2022 at 11:59pm on gradescope

Question 1. Let $X \sim \text{Binomial}(n, p)$.

(a) Compute the moment generating function $M_X(t)$ of X.

(b) Compute $E[X^2]$ using the moment generating function from (a).

a) We know that $X = \sum_{i=1}^n X_i$ where $X_i \sim \text{Ber}(p)$ and the X_i are independent. By properties of moment generating functions we have

$$M_X(t) = M_{X_1}(t) \cdots M_{X_n}(t) = (M_{X_1}(t))^n.$$

For a Bernoulli random variable we have

$$M_{X_1}(t) = E[e^{tX_1}] = 1P(X_1 = 0) + e^t P(X_1 = 1) = (1-p) + pe^t.$$

Therefore

$$M_X(t) = ((1-p) + pe^t)^n.$$

b) We have

$$\begin{aligned} E[X^2] &= M_X''(0) \\ &= \frac{d^2}{dt^2} \left(((1-p) + pe^t)^n \right) \big|_{t=0} \\ &= \frac{d}{dt} \left(n((1-p) + pe^t)^{n-1} pe^t \right) \big|_{t=0} \\ &= \left(n(n-1)((1-p) + pe^t)^{n-2} p^2 e^{2t} + n((1-p) + pe^t)^{n-1} pe^t \right) \big|_{t=0} \\ &= n(n-1)p^2 + np \end{aligned}$$

Question 2. Let X be uniformly distributed on $[0, 1]$. Find the moment generating function for X.

Solution. We obtain $\int_0^1 e^{tx}\, dx = \frac{e^t - 1}{t}$.