# Math 201, Spring 2022

## Problem Set # 11 Solutions

Due April 26, 2022 at 11:59pm on gradescope

**Question 1.** Suppose that a professor chooses a random student in a class of 40 students (there are 23 girls and 17 boys in the class) to perform a calculation on the board. The professor repeats this procedure 3 times, choosing a new student each time (i.e. no student will go twice). Let $X$ be the total number of boys chosen. Calculate the mean and variance of $X$. (You will want to use a calculator for this one.)

**Hint 1:** Use indicator random variables

$$X_i = \begin{cases} 1 & \text{if a boy is chosen for calculation } i \\ 0 & \text{else.} \end{cases}$$

**Hint 2:** Notice that the distribution of $X_i$ does not depend on $i$.

We use indicator random variables

$$X_i = \begin{cases} 1 & \text{if a boy is chosen for calculation } i \\ 0 & \text{else.} \end{cases}$$

Then $X = \sum_{i=1}^{3} X_i$ and we have

$$E[X] = E\left[\sum_{i=1}^{3} X_i\right] = \sum_{i=1}^{3} E[X_i].$$

Since the $X_i$ are indicator random variables

$$E[X_i] = P(X_i = 1) = 3/40$$

and so

$$E[X] = 3 \cdot 17/40 = 51/40.$$

*** For the variance we have

$$\begin{aligned} \text{Var}(X) &= \text{Var}\left(\sum_{i=1}^{3} X_i\right) \\ &= \sum_{i=1}^{3} \text{Var}(X_i) + 2 \sum_{1 \le i < j \le 3} \text{Cov}(X_i, X_j) \end{aligned}$$

For each $i$, we have

$$\text{Var}(X_i) = E[X_i]^2 - E[X_i]^2 = P(X_i = 1) - P(X_i = 1)^2 = 17/40 - (17/40)^2 = 391/1600 = 0.24438.$$

For the covariance with $i \neq j$, we have

$$\begin{aligned}\text{Cov}(X_i, X_j) &= E[X_i, X_j] - E[X_i]E[X_j] \\ &= P(X_i = 1, X_j = 1) - P(X_i = 1)P(X_j = 1) \\ &= (17/40)(16/39) - (17/40)^2 = -\frac{391}{62400} = -0.006266\end{aligned}$$

We find

$$\begin{aligned}\text{Var}(X) &= 3\text{Var}(X_1) + 2\binom{3}{2}\text{Cov}(X_1, X_2) \\ &= 3 \cdot \frac{391}{1600} + 6 \cdot \left(-\frac{391}{62400}\right) \\ &= .6955\end{aligned}$$

**Question 2.** (a) A fair die is rolled until three different numbers are seen. Let $X$ be the number of rolls this requires. Find $E[X]$ and $\text{Var}(X)$. [Hint: Use the technique of the coupon collector problem in the book.]

(b) A fair coin is flipped 12 times. Find the expected value for the number of times you see three consecutive tails.

**Solutions.**

(a) The first number seen is on the first roll. If $W_1$ is the number of rolls til a second number is seen, then $W_1 \sim \text{Geom}(5/6)$ since each the die is rolled there is a 5/6 chance something different from the first number appears. Likewise the number of rolls between seeing the second number and the third is $W_2 \sim \text{Geom}(4/6)$. We have

$$E[X] = 1 + E[W_1] + E[W_2] = 1 + 6/5 + 6/4 = 37/10 = 3.7.$$

Since Since $W_1$ and $W_2$ are independent, we also have $\text{Var}(X) = \text{Var}(W_1) + \text{Var}(W_2)$, which gives

$$\text{Var}(X) = \frac{1-5/6}{(5/6)^2}\frac{1-4/6}{(4/6)^2} = .99$$

(b) For me, the usual interpretation of this is that this includes four consecutive tails as well, so that if you had say $TTTTHTHTHTHT$, that would count as two runs of three. Done this way, we have ten indicator variables $X_i$, each being 1 if there run of three tails starting on the $i$-th flip and 0 otherwise. Then each $X_i$ has expectation 1/8. Adding up, we get 10/8.

A student showed me an example in the book where they do this problem and think of four consecutive tails as not being three consecutive tails. Done this way, we still have ten indicator variables $X_i$. But note that for $2 = 1, ...9$, we must check that the $i-1$ flip and the $i+3$ flip are both heads. This happens with chance $1/32$. For the $X_{10}$, there is no $i+3$ flip and for $i = 1$, there is no $i-1$ flip so we get $1/16$. Done this way the answer is then $\frac{8}{32} + \frac{2}{16} = \frac{12}{32} = \frac{3}{8}$.