Math 201, Spring 2022

Problem Set # 10

Due April 16, 2022 at 11:59pm on gradescope

Question 1. Suppose that $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(r)$ are independent. Find the probability $P(X < Y)$.

We can decompose the event $\{X < Y\} = \cup_{k=1}^{\infty}\{X < k, Y = k\}$. Since these are disjoint events we have

$$\begin{aligned}
P(X < Y) &= \sum_{k=1}^{\infty} P(X < k)P(Y = k) \\
&= \sum_{k=1}^{\infty}(1-(1-p)^{k-1})(1-r)^{k-1}r \\
&= r\sum_{k=1}^{\infty}(1-r)^{k-1} - r\sum_{k=1}^{\infty}(1-p)^{k-1}(1-r)^{k-1} \\
&= \frac{r}{1-(1-r)} - \frac{r}{1-(1-p)(1-r)} \\
&= 1 - \frac{r}{1-(1-p)(1-r)} \\
&= \frac{p(1-r)}{r+p(1-r)} \\
&= \frac{1}{1+\frac{r}{p(1-r)}}
\end{aligned}$$

Question 2. Let $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\lambda)$ be independent random variables. Let $t > 0$. Let A be the event that $X < Y$. Let B be the event that $\min(X, Y) > t$.

(a) Find $P(A \cap B)$. (That is, find the change that $X < Y$ and $\min(X, Y) > t$). You may leave powers of e in your answer.

(b) Are A and B independent? (That is, do we have $P(A \cap B) = P(A)P(B)$?) You may use the fact that if $T = \min(X, Y)$, then $T \sim \text{Exp}(\lambda + \lambda)$, proved in the book in Example 6.34. You may also use the formula proved in Example 6.33 for$P(X < Y)$.

Solutions.

(a) The region corresponding to $A \cap B$ is simply $X \geq t, y \geq X$, so

$$P(A \cap B) = \int_t^{\infty}\int_x^{\infty} \lambda\lambda e^{-\lambda x}e^{\lambda y}\,dx\,dy = \int_t^{\infty} \lambda e^{-\lambda x}e^{-\lambda x}\,dx = \frac{1}{2}e^{-(2\lambda)t}$$

(b) We have $P(A) = \frac{\lambda}{\lambda+\lambda} = \frac{1}{2}$ by Example 6.33 (you can also simply reason this by noting that the two variables have the same distribution so the chance of one being greater than the other must be .5). And we have $P(B) = e^{-(\lambda+\lambda)t}$ by the usual formula for the CDF of an exponential. So, yes, $P(A \cap B) = P(A)P(B)$, which means that A and B are independent.