Math 201, Spring 2021

Problem Set # 1

Due February 12, 2021 at 11:59pm on gradescope

Question 1. Eight rooks are placed randomly on a chess board. What is the probability that none of the rooks can capture any of the other rooks? Translation for those who are not familiar with chess: pick 8 unit squares at random from an 8x8 grid. What is the probability that no two chosen squares share a row or a column?

Solution:

There are $\binom{64}{8}$ ways to place the rooks on the chess board. Since no rooks can capture any other rooks, each rook must be in its own row. There are 8 different columns the first rook can be placed in, 7 the second rook can be placed in and so on. There are 8! different ways to place the rooks so the probability is

$$\frac{8!}{\binom{64}{8}} \approx 9.11 \cdot 10^{-6}$$

Question 2. Show that it is not possible to choose a uniform positive integer at random. (In other words, we cannot define a probability measure on the positive integers that can be considered uniform.)

Hint: What would be the probability of choosing a particular number?

Solution: Suppose the probability of choosing the number 2 is c. Since c is a probability we must have $c \geq 0$. Since probabilities are uniform the probability of choosing any other integer is c as well. We can decompose the sample space as a union of the singletons,

$$\Omega = \{1\} \cup \{2\} \cup \{3\} \cdots = \cup_{i=1}^{\infty} \{i\}.$$

Since this is a countable set of pairwise disjoint events we must have

$$\begin{aligned} P(\Omega) &= P\left(\cup_{i=1}^{\infty} \{i\}\right) \\ &= \sum_{i=1}^{\infty} P(\{i\}) = \sum_{i=1}^{\infty} c. \end{aligned}$$

Since P is a probability measure we must have $P(\Omega) = 1$. However if $c > 0$ then $\sum_{i=1}^{\infty} c = \infty$ and if $c = 0$ then $\sum_{i=1}^{\infty} c = 0$. In either case we have a contradiction which implies that we cannot define a uniform probability measure on the positive integers.