MTH 201

Midterm 1

March 12, 2020

Name: Key

UR ID: ______________________

Circle your Instructor's Name:

Ian Alevy Thomas Tucker

Instructions:

- The presence of calculators, cell phones, and other electronic devices (other than a device for being on Zoom during the exam and uploading your exam afterwards) at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.
- In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$ and you can leave your answers in terms of $\binom{n}{k}$ or $k!$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$. Summations must also be evaluated, in particular, the symbols "$\sum$" or "$\cdots$" should not appear in final answers.
- This exam is out of 50 points. You are responsible for checking that this exam has all 10 pages.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE: ______________________

1. (5 points) For this problem, justification is not required and partial credit will **not** be awarded.

Suppose an urn contains 5 red balls, 7 green balls and 9 orange balls. Five balls are drawn randomly one at a time without replacement from the urn.

(a) 2 pt: What is the probability that the sample contains exactly 3 red balls?

$$\frac{\binom{5}{3}\binom{16}{2}}{\binom{21}{5}}$$

(b) 3 pts: What is the probability that the sample contains at least one red ball **and** one green ball?

$$P(\{\text{at least } 1\, r\} \cap \{\text{at least } 1\, g\}) = 1 - P(\{\text{at least } 1\, r\}^c \cup \{\text{at least } 1\, g\}^c)$$

$$= 1 - (P(\text{no red}) + P(\text{no green}) - P(\{\text{no red}\} \cap \{\text{no green}\}))$$

$$= 1 - \frac{\binom{16}{5} + \binom{14}{5} - \binom{9}{5}}{\binom{21}{5}}$$

2. (8 points) Peter and Mary take turns rolling a fair die. If Peter rolls 1 he wins and the game stops. If Mary rolls 3 or 6, she wins and the game stops. They keep rolling in turn until one of them wins. Suppose Peter rolls first.

(a) 2 pts: What is the probability that Mary wins on her fifth roll?

$$\left(\frac{5}{6}\right)^5\left(\frac{4}{6}\right)^4\left(\frac{2}{6}\right) = \left(\frac{5}{6}\right)^5\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)$$

(b) 6 pts: What is the probability that Mary wins? (To receive full credit, you must evaluate any infinite series in your answer.)

$$P(\text{Mary wins}) = \frac{5}{6}\left(\frac{2}{6}\right)+\left(\frac{5}{6}\right)^2\left(\frac{4}{6}\right)\left(\frac{2}{6}\right)+\left(\frac{5}{6}\right)^3\left(\frac{4}{6}\right)^2\left(\frac{2}{6}\right)+\ldots$$

$$= \frac{5}{18}\sum_{k=0}^{\infty}\left(\frac{5}{9}\right)^k = \frac{5}{18}\,\frac{1}{1-\frac{5}{9}} = \frac{5}{18}\times\frac{9}{4}$$

$$= \boxed{\frac{5}{8}}$$

3. (8 points) Consider events A, B, and C which are mutually independent (recall that this means that A and B are independent, A and C are independent, B and C are independent, and that $P(A \cap B \cap C) = P(A)P(B)P(C)$) with $P(A) = 1/2$, $P(B) = 1/4$ and $P(C) = 1/2$.

(a) 3 pts: Compute $P(A \cup B)$.

$$P(A \cup B) = P(A) + P(B) - P(A \wedge B) = P(A) + P(B) - P(A)P(B)$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$$

(b) 5 pts: Are the events $A \cup B$ and C independent? Explain your answer carefully or no credit will be given.

$$P((A \cup B) \wedge C) = P((A \wedge C) \cup (B \wedge C))$$

$$= P(A \wedge C) + P(B \wedge C) - P(A \wedge B \wedge C)$$

$$= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)$$

$$= \frac{1}{4} + \frac{1}{8} - \frac{1}{16} = \frac{5}{16}$$

$$P(A \cup B)P(C) = \frac{5}{8} \cdot \frac{1}{2} = \frac{5}{16}$$

yes they are independent

4. (8 points) A fair coin is flipped four times. Let A be the event that tails comes up at least three times. Let B be the event that the first three flips are tails.

(a) 3 pts: Find $P(A)$.

$$P(A) = P(\text{exactly } 3\ T) + P(4\ T)$$

$$= \binom{4}{3}\frac{1}{2^4} + \frac{1}{2^4} = \frac{1}{16}(4+1) = \boxed{\frac{5}{16}}$$

(b) 5 pts: Find the conditional probability $P(B \mid A)$.

$$P(B|A) = \frac{P(B \wedge A)}{P(A)} = \frac{1/8}{5/16} = \boxed{\frac{2}{5}}$$

$$P(A) = 5/16$$

$$P(B \wedge A) = P(B) = \frac{1}{8}$$

5. (8 points)

The continuous random variable X is uniformly distributed on the interval $[-2, 5]$.

(a) 4 pts: Find $P(X < 0)$.

$$\boxed{\frac{2}{7}}$$

(b) 4 pts: Find $P(X^2 > 1)$.

$$P(X^2 > 1) = P(X > 1) + P(X < -1)$$

$$= \frac{4}{7} + \frac{1}{7} = \boxed{\frac{5}{7}}$$

6. (6 points)

Three numbers are chosen with replacement from the set $\{1, 2, 3\}$.

(a) 2 pts: Find the chance that no number is chosen twice.

$$\frac{3!}{3^3} = \frac{2}{9}$$

(b) 4 pts: Find the chance that at least two different numbers are chosen.

~~P(exactly 2 different numbers drawn) = 3×2×1 + 3×1×2 +~~

$$P(\text{all numbers the same}) = \frac{3}{3^3} = \frac{1}{9}$$

$$P(\text{at least 2 different}) = 1 - P(\text{all same}) = 1 - \frac{1}{9} = \boxed{\frac{8}{9}}$$

7. (7 points) There are three types of coins in circulation. There are fair coins with $P(H) = 1/2$, moderately biased coins with $P(H) = 1/3$ and heavily biased coins with $P(H) = 1/5$. Suppose 1/2 of the coins are fair, 1/4 are moderately biased and 1/4 are heavily biased. A coin is flipped twice and the outcome is heads followed by tails. What is the probability that the coin is fair? **You may leave your answer as a fraction.**

$$P(\text{fair} \mid HT) = \frac{P(HT \mid \text{fair})\, P(\text{fair})}{P(HT \mid \text{fair})\, P(\text{fair}) + P(HT \mid \text{moderately biased})\, P(\text{moderately biased}) + P(HT \mid B)\, P(B)}$$

$$= \frac{1/4 \times 1/2}{\frac{1}{4} \times \frac{1}{2} + \frac{1}{3} \times \frac{2}{3} \times \frac{1}{4} + \frac{1}{5} \times \frac{4}{5} \times \frac{1}{4}}$$

$$= \boxed{\frac{1/2}{1/2 + \frac{2}{9} + 4/25}} = \frac{225}{397} \simeq 0.567$$