MTH 201

Midterm 1

February 22, 2022

Name: ______________________________

UR ID: ______________________________

Circle your Instructor's Name:

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Instructions:

- The presence of calculators, cell phones, and other electronic devices (other than a device for being on Zoom during the exam and uploading your exam afterwards) at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.
- In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$ and you can leave your answers in terms of $\binom{n}{k}$ or $k!$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$. Summations must also be evaluated, in particular, the symbols "$\sum$" or "$\cdots$" should not appear in final answers.
- This exam is out of 50 points. You are responsible for checking that this exam has all 9 pages.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

QUESTION	VALUE	SCORE
1	5	
2	8	
3	7	
4	8	
5	8	
6	8	
7	6	
TOTAL	50	

1. (5 points) For this problem, justification is not required and partial credit will **not** be awarded.

Suppose an urn contains 3 red balls, 4 green balls, and 5 orange balls. Five balls are drawn randomly one at a time without replacement from the urn.

(a) What is the probability that the sample contains exactly 1 red ball?

12 balls total

5 drawn

3 R

9 G and O

Answer: $\binom{3}{1}\binom{9}{4} \Big/ \binom{12}{5}$

(b) What is the probability that the sample contains exactly one green ball and exactly two orange balls?

This also means exactly 2 red

Answer: $\dfrac{\binom{4}{1}\binom{5}{2}\binom{3}{2}}{\binom{12}{5}}$

2. (8 points)

Peter rolls a die repeatedly. If he rolls a 5 or a 6, he wins and the game ends. If he rolls a 1 he loses and the game ends. If he rolls a 2, 3, or 4, then the game continues and he rolls again.

(a) What is the probability that Peter wins on his third roll?

Each roll he has $1/3$ chance of winning

$1/6$ losing

$1/2$ continuing

so $\left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)$

Answer: $\left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right) = 1/12$

(b) What is the probability that Peter eventually wins (to receive full credit, you must evaluate any infinite sums that appear)?

Chance he wins on kth roll is

$\left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{3}$ so

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{3} = \left(\sum_{r=0}^{\infty} (1/2)^r\right) \cdot \frac{1}{3} = 2/3$$

Answer: $2/3$

3. (7 points) We roll two fair, 3-sided die (each of the numbers 1, 2, and 3 are equally likely to appear on each roll) and record the outcome of each die. Let A be the event that the first roll is an odd number and B be the event that the sum of the outcomes was even.

(a) Compute $P(A)$ and $P(B)$.

$2\ (1,1)\quad 3\ (2,1)\quad 4\ (3,1)$

$3\ (1,2)\quad 4\ (2,2)\quad 5\ (3,2)$

$4\ (1,3)\quad 5\ (2,3)\quad 6\ (3,3)$

$P(A) = 2/3$

$P(B) = 5/9$

Answer: $P(A) = 2/3 \quad P(B) = 5/9$

(b) Are the events A and B independent? Justify your answer using probabilities.

$A \cap B = \{(1,1), (1,3), (3,1), (3,3)\}$

$P(A \cap B) = 4/9 \neq \underbrace{\frac{2}{3}}_{P(A)} \cdot \underbrace{\frac{5}{9}}_{P(B)}$

Answer: Not independent

4. (8 points) An urn contains 4 red balls and 6 black balls. Consider the following game. A player rolls a fair (6-sided) die. If he rolls 3 or less, he loses immediately. Otherwise he selects, at random and without replacement, as many balls from the urn as the number that came up on the die. The player wins if all four red balls are among the selected balls.

(a) Compute the winning probability for this game.

W - winning

X # rolled

$$P(W \mid X < 4) = 0$$

$$P(W \mid X = 4) = \frac{1}{\binom{10}{4}}$$

$$P(W \mid X = 5) = \frac{6}{\binom{10}{5}}$$

$$P(W \mid X = 6) = \frac{\binom{6}{2}}{\binom{10}{6}}$$

Answer:

$$\frac{1}{6} \cdot \frac{1}{\binom{10}{4}} + \frac{1}{6} \cdot \frac{6}{\binom{10}{5}} + \frac{1}{6} \frac{\binom{6}{2}}{\binom{10}{6}}$$

(b) Jane tells you that she recently played this game once and won. What is the probability that she rolled a 6?

$$P(X=6 \mid W)$$

$$= \frac{P(W, X=6)}{P(W)}$$

$$= \frac{\frac{1}{6} \cdot \frac{\binom{6}{2}}{\binom{10}{6}}}{\frac{1}{6}\frac{1}{\binom{10}{4}} + \frac{1}{6}\frac{6}{\binom{10}{5}} + \frac{1}{6}\frac{\binom{6}{2}}{\binom{10}{6}}}$$

Answer:

$$\frac{\binom{6}{2}}{\binom{10}{6}} \Big/ \left(\frac{1}{\binom{10}{4}} + \frac{6}{\binom{10}{5}} + \frac{\binom{6}{2}}{\binom{10}{6}}\right) \quad)$$

5. (8 points)

The continuous random variable Z is uniformly distributed on the disc of radius 3 inches (that is the set of all (x, y) such that $x^2 + y^2 \leq 9$). Find the chance that

(a) Find the chance that Z is within 1 inch of the origin $(0, 0)$.

$\frac{\pi}{9\pi} = 1/9$

Answer: $1/9$

(b) Find the chance Z is two or more inches away from the origin $(0, 0)$.

$\frac{9\pi - 4\pi}{9\pi} = 5/9$

Answer: $5/9$

6. (8 points)

Four balls are chosen **with replacement** from an urn containing one red ball, one green ball, and one yellow ball.

(a) Find the chance that all four balls chosen are the same color.

For each color, you have

a $\left(\frac{1}{3}\right)^4$ chance

so $3 \cdot \left(\frac{1}{3}\right)^4 = \frac{1}{27}$

Answer: 1/27

(b) Find the chance that there is at least one color ball that is never chosen.

A - no Red

B - no Green

C - no yellow

$p(A) = p(B) = p(C) = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

$p(A \cap B) = p(A \cap C) = p(B \cap C) = \left(\frac{1}{3}\right)^4$

$p(A \cap B \cap C) = 0$

so

$p(A \cup B \cup C) = 3 \cdot \frac{16}{81} - 3 \frac{1}{81} - 0$

$= \frac{45}{81} = \frac{5}{9}$

Answer: 5/9

7. (6 points) A (unfair) coin comes up tails with probability .2. A number k is picked at random from the set $\{1, 2, 3\}$ and then the coin is flipped k times. Let X be the number of tails that appear. Write down the probability mass function for X.

We use a 2-stage process w/ binomial distribution cet 2nd stage

Answer:

$$p(X=0) = \frac{1}{3} \cdot .8 + \frac{1}{3}(.8)^2 + \frac{1}{3}(.8)^3$$

$$p(X=1) = \frac{1}{3} \cdot .2 + \frac{1}{3} \cdot \binom{2}{1}(.8 \cdot .2) + \frac{1}{3}\binom{3}{1} .2 \cdot (.8)^2$$

$$p(X=2) = \frac{1}{3} \cdot (.2)^2 + \frac{1}{3} \cdot \binom{3}{2}(.2)^2 \cdot .8$$

$$p(X=3) = \frac{1}{3}(.2)^3$$