Math 201: Introduction to Probability

Final Exam

May 6, 2019

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Instructions:

1. Indicate your instructor with a check in the appropriate box:

Krishnan	MW 14:00	
Tucker	MW 10:25	

2. Read the notes below:

   - **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
   - Notes of any kind are strictly forbidden.
   - The table for the normal distribution is on last page of the exam.
   - **Put your FINAL answers in the boxes provided.**
   - **You must show your work to get full credit, but do not put it in the final answer box.**
   - You do not need to simplify complicated numerical expressions such as $\binom{100}{30}$ and 50! to a number.
   - You are responsible for checking that this exam has all 31 pages.

3. Read the following Academic Honesty Statement and sign:

   I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:______________________________

Formulae If $X \sim \text{Binomial}(n, p)$, then

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, \ldots, n$$

$$E[X] = np, \text{Var}(X) = np(1-p), M(t) = (1 - p + pe^t)^n$$

If $X \sim \text{Geometric}(p)$, then

$$P(X = k) = (1-p)^{k-1} p \quad k = 1, 2, \ldots$$

$$E[X] = 1/p, \text{Var}(X) = (1-p)/p, M(t) = pe^t/(1-(1-p)e^t) \quad t < -\log(1-p)$$

On a related note, the sum of a geometric series is $1 + x + x^2 + \cdots = 1/(1-x), |x| < 1$.
If $X \sim \text{Exp}(\lambda)$,

$$f(x) = \lambda e^{-\lambda x} \quad x > 0, E[X] = \frac{1}{\lambda} \quad \text{Var}(X) = \lambda^{-2} \quad M(t) = \lambda/(\lambda - t) \quad t < \lambda$$

If $X \sim \text{Poisson}(\lambda)$, then

$$p(k) = e^{-\lambda} \lambda^k / k! \qquad k = 0, 1, \ldots$$

$$E[X] = \lambda \quad \text{Var}(X) = \lambda \quad M(t) = \exp(\lambda(e^t - 1))$$

If $X \sim N(0, 1)$ is standard normal, $\Phi(t) = P(X \leq t)$, and

$$E[X] = 0, \text{Var}(X) = 1, \quad f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad M(t) = e^{t^2/2}$$

The cdf of a random variable X is $F(t) = P(X \leq t)$. If X is continuous and has pdf $f(x)$,

$$F(t) = \int_{-\infty}^{t} f(x) dx \quad f(t) = \frac{d}{dt} F(t)$$

The expectation of continuous and discrete random variables may be written as

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx, \quad E[X] = \sum_{i \in \text{ran}(X)} i P(X = i)$$

To obtain a confidence interval with probability $x\%$, use

$$x/100 = P(|p - \hat{p}| \leq \epsilon) \geq 2\Phi(2\epsilon\sqrt{n}) - 1$$

The moment generating functions for continuous and discrete random variables are:

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx, \quad M(t) = \sum_{i \in \text{ran}(X)} e^{ti} P(X = i)$$

The covariance of a pair of random variables is

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

The correlation coefficient is

$$\rho = \frac{E[XY] - E[X]E[Y]}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$$

Part A		
QUESTION	VALUE	SCORE
1	10	
2	12	
3	12	
4	16	
5	10	
6	15	
TOTAL	75	

Part B		
QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	15	
5	10	
6	20	
7	10	
8	15	
TOTAL	100	

Part A

1. (10 points) An elementary class consists of four girls and two boys (for a total of six students). Two students are chosen (without replacement) from the class.

(a) What is the chance that exactly one boy and one girl is chosen?

$$\frac{\binom{4}{1}\binom{2}{1}}{\binom{6}{2}}$$

Answer:

$$\frac{8}{15}$$

(b) What is the chance that two boys are chosen?

$$\frac{\binom{2}{2}}{\binom{6}{2}}$$

Answer:

$$\frac{1}{15}$$

2. (12 points)

Suppose that $X \sim$ Geometric(1/3), that is $P(X = k) = \left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right)$ for positive integers k and $P(X = \ell) = 0$ for all ℓ that are not positive integers.

To receive full credit on the questions below, you must evaluate any infinite sums.

(a) Find $P(X > 1)$.

$$= 1 - P(X \leq 1)$$

$$= 1 - P(X = 1) = 1 - \frac{1}{3} = \frac{2}{3}$$

Some of you directly used the fact that

$P(X > k) = (1-p)^k$ (If your 1st success happens after the k^{th} trial, the 1st k trials must be failures and vice-versa)

$$\Rightarrow P(X > 1) = \left(\frac{2}{3}\right)^1$$

Answer:

$$P(X > 1) = \frac{2}{3}$$

(b) Find $P(\text{X is odd})$.

$$P(X \text{ is odd}) = P(X=1) + P(X=3) + \cdots$$

$$= \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)$$

$$= \left(\frac{1}{3}\right) \sum_{i=0}^{\infty} \left[\left(\frac{2}{3}\right)^2\right]^i = \left(\frac{1}{3}\right) \frac{1}{1 - \frac{4}{9}}$$

$$= \frac{1}{3} \cdot \frac{9}{5} = \frac{3}{5}$$

Answer:

$$\frac{3}{5}$$

3. (12 points)

A bowl is filled with jelly beans made by three different companies. There are 50 beans made by Company A, 40 by Company B, and 10 by Company C. Ten percent of the beans made by Company A are red, 20 percent of the beans made by Company B are red, and 30 percent of the beans made by Company C are red.

(a) What is the probability that a randomly selected bean from the bowl is red?

$$\frac{\#\text{ of red beans}}{\text{Total \# of beans}} = \frac{(0.1)50 + (0.2)40 + (0.3)10}{100} = \frac{16}{100}$$

Answer:

$$P(R) = \frac{16}{100}$$

(b) If a red bean is randomly selected from the bowl, what is the chance that bean is made by Company A?

R = Red bean selected.

A = Bean made by company A

$$P(A \mid R) = \frac{P(A \cap R)}{P(R)} = \frac{P(R \mid A)\,P(A)}{P(R)}$$

Answer:

$$\frac{(0.1)(0.5)}{0.16} = \frac{5}{16}$$

4. (16 points)

A fair coin is flipped four times. The random variable X is the number of tails that appear in the first two flips and the random variable Y is the number of tails that appear in the final three flips.

(a) Find $E[X+Y]$.

$$E[X] + E[Y] = E[X+Y]$$

$$E[X] = 2 \cdot \frac{1}{2} \qquad E[Y] = 3 \cdot \frac{1}{2}$$

$X \sim Bin(2, \frac{1}{2})$

$Y \sim Bin(3, \frac{1}{2})$

Partial credit for linearity

Full credit for final answer.

Answer:

$$\frac{5}{2}$$

(b) Find $P(Y = 2)$.

$$P(Y=2) = \binom{3}{2}\left(\frac{1}{2}\right)^3$$

Partial credit for realizing $Y \sim Bin(3, \frac{1}{2})$

Answer:

$$\frac{3}{8}$$

(c) Find the conditional probability $P(Y = 2 \mid X = 0)$.

2 points for noting that if $X = 0$, then the 2nd flip must be heads.

Full credit if you notice that # of tails in the remaining 2 flips is $\text{Bin}(2, \frac{1}{2}) \Rightarrow P(Y=2 \mid X=0) = \binom{2}{2}\left(\frac{1}{2}\right)^2$

Answer:

$$\frac{1}{4}$$

(d) Are X and Y independent? Give a mathematical reason.

Partial credit for arguing correctly without mathematical reasoning

Answer:

NOT INDEP.

$P(Y=2 \mid X=0) \neq P(Y=2)$

5. (10 points) Suppose that we have a random variable X such that

$$P(X \leq t) = \begin{cases} 0; \ t < 0 \\ t^2; \ 0 \leq t \leq 1 \\ 1; \ t > 1 \end{cases}$$

Find $E[X]$.

$$E[X] = \int_{-\infty}^{\infty} f(x)\, x \, dx = \int_0^1 2x^2 \, dx = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$f(x) = \frac{d}{dx} P(X \leq x)$$

$$= \begin{cases} 0 & x < 0 \\ 2x & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

Answer:

$\frac{2}{3}$

6. (15 points) A fair six-sided die is rolled twice. We let X denote the number obtained on the first roll, and Y the number obtained on the second roll.

(a) Find $P(\min(X, Y) = 2)$.

$$\{\min(X,Y)=2\} = \left\{\begin{matrix} (2,2), \dots & (2,6) \\ (3,2)\ (4,3), \dots & \end{matrix}\right\} \Rightarrow P(\min(X,Y)=2) = \frac{9}{36}$$

Alternately: $P(\min(X,Y)=2) = P(\min(X,Y)>1) - P(\min(X,Y)>2)$

$$= P(X>1)P(Y>1) - P(X>2)P(Y>2) = \left(\frac{5}{6}\right)^2 - \left(\frac{4}{6}\right)^2 = \frac{25-16}{36}$$

Answer:

$$\frac{9}{36}$$

Partial credit for getting the numerator right.

(b) Find $P(X > Y)$.

$$= \sum_{i=1}^{5} P(X>i \mid Y=i)\,P(Y=i)$$

Law of total probability

$$= \sum_{i=1}^{5} \left(\frac{6-i}{6}\right)\frac{1}{6} = \frac{1}{36}\left[\sum_{i=1}^{5} 6 - \sum_{i=1}^{5} i\right]$$

$$= \frac{1}{36}\left[30 - \frac{6\cdot 5}{2}\right] = \frac{15}{36}$$

$$X>Y \quad \begin{bmatrix} \cdot & & & & & \\ (2,1) & \cdot & & & & \\ (3,1) & (3,2) & \cdot & & & \\ (4,1) & (4,2) & (4,3) & \cdot & & \\ \vdots & & & & \cdot & \\ & & & & & \cdot \end{bmatrix}$$

$$= \frac{5\cdot 6}{2} = 15$$

Answer:

$$\frac{15}{36}$$

(c) Find the chance that $X+Y$ is odd.

If $X+Y$ is ~~odd~~ odd then X is odd, Y is even. or
X is even, Y is odd or

$$P(X+Y \text{ is odd}) = P(X \text{ is odd})P(Y \text{ is even}) + P(X \text{ is even})P(Y \text{ is odd})$$

using independence and disjointness.

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

Alternately: $\{ (1,2), (1,4), (1,6)$
$(2,1), (2,3), (2,5)$
$\vdots$
$\}$

has cardinality $6 \cdot 3 = 18$

Partial credit for listing correct pairs.

Partial credit for a good argument.

Answer: $\frac{1}{2}$

Part B

1. (10 points) Let X be a random variable with distribution $X \sim Exp(\lambda)$. Suppose the lifetime of a certain device (measured in months), is given by $Y = \sqrt{X}$.

(a) Compute the cumulative distribution function for Y.

Hint: Start with $P(Y \leq t)$ and recall the cumulative distribution function of X.

$$P(Y \leq t) = P(\sqrt{X} \leq t) = P(X \leq t^2)$$

$$= 1 - e^{-\lambda t^2}$$

Answer:

$$1 - e^{-\lambda t^2}$$

(b) Compute the probability density function of Y.

$$\frac{d}{dt} P(Y \leq t) = 2\lambda t\, e^{-\lambda t^2}$$

Answer:

$$2\lambda t\, e^{-\lambda t^2}$$

(c) Find the probability the device lasts at least 6 months.

$$P(Y > 6) = 1 - P(Y \le 6)$$
$$= e^{-\lambda 6^2}$$

Answer:

$$e^{-6^2 \lambda}$$

2. (10 points) In each of the following questions identify the described random variable as either Bernoulli, Binomial, Poisson, Geometric, or Exponential and describe a reasonable numerical value for all its parameters. Please use the conventional notation for the parameters i.e., p, λ, n, etc. No explanation is necessary for this problem. Only give your answers.

(a) Two 6-sided dice are simultaneously rolled over and over again. Each time the product of the dice is 6 or 12, someone gives you 1 dollar. Let X_1 be the amount of money you have received after 50 rolls. What type of random variable is X_1 and what are its parameters?

Answer:

$X_1 \sim \text{Binomial}\left(\frac{2}{9}\right)$

(b) Lola loves to play basketball in her driveway. On average, she scores a basket about 3 out of 5 attempts. Her mother calls her in for dinner, but Lola is determined to score one more basket before going inside. Let X_2 be the number of attempts before she comes inside. What type of random variable is X_2 and what are its parameters?

Answer:

$X_2 \sim \text{Geom}\left(\frac{3}{5}\right)$

(c) Suppose that X_3 is a continous random variable with mean equal to 2 with the property that for any real numbers s and t we have $P(X_3 > s) = P(X_3 > t + s \mid X_3 > t)$ (this is sometimes called the "memoryless" poperty). What type of random variable is X_3 and what are its parameters?

Answer:

$X_3 \sim \text{Exp}\left(\frac{1}{2}\right)$

(d) Suppose X_4 represents the number of people coming into a store on a certain day. On average the owner sees 5 customers per day. What type of random variable would you use to model X_4 and what are its parameters?

Answer:

$X_4 \sim \text{Poisson}(5)$

3. (10 points) Suppose the moment generating function of a random variable is of the form

$$M_Z(t) = \left(\frac{1}{2}e^{-t} + \frac{1}{3}e^{3t} + \frac{1}{6}e^{7t}\right)^{50}$$

(a) Write Z as a sum of independent identically distributed random variables. Give the pdf or pmf of the random variables involved in the sum.

$Z = Y_1 + Y_2 + \cdots + Y_{50}$] 2 pts (if you have 50 rvs)

$M_Z(t) = M_{Y_1}(t) M_{Y_2}(t) \cdots M_{Y_{50}}(t) = (M_{Y_1}(t))^{50}$

$M_{Y_1}(t) = \frac{1}{2}e^{-t} + \frac{1}{3}e^{3t} + \frac{1}{6}e^{7t}$

$p_{Y_1}(-1) = \frac{1}{2}$

$p_{Y_1}(3) = \frac{1}{3}$

$p_{Y_1}(7) = \frac{1}{6}$

4 pts if you get the PMF right.

(b) Find $E[Z]$.

Method 1

$$E[Z] = \overbrace{50}^{2\text{ pts}}\,\overbrace{E[Y_1]}^{2\text{ pts}}$$

$$E[Y_1] = -1 \cdot \frac{1}{2} + \frac{1}{3} \cdot 3 + \frac{1}{6} \cdot 7 = \frac{-6+12+14}{12}$$

$$= \frac{20}{12} = \frac{5}{3}$$

$$E[Z] = \frac{250}{3}$$

Method 2

$$\frac{d}{dz} M_Z(t)\Big|_{t=0} = 50\left(\frac{1}{2}e^{-t} + \frac{1}{3}e^{3t} + \frac{1}{6}e^{7t}\right)^{49} \cdot \left(\frac{-1}{2}e^{-t} + \frac{3}{3}e^{3t} + \frac{7e^{7t}}{6}\right)\Big|_{t=0}$$

$$= 50 \cdot 1^{49}\left(-\frac{1}{2} + \frac{3}{3} + \frac{7}{6}\right)$$

$$= 50 \cdot \frac{5}{3}$$

Answer:

$$\frac{250}{3}$$

Part B

1. (15 points) Let Let X and Y be random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} Cx^2y, & \text{if} \quad x \in [0,1], y \in [0,2] \\ 0 & \text{else.} \end{cases}$$

where $C > 0$ is a constant.

a) Find the constant C.

$$\int_0^2\int_0^1 Cx^2y\,dx\,dy = C\int_0^2 y\left[\frac{x^3}{3}\right]_0^1 dy = \frac{C}{3}\int_0^2 y\,dy$$

$$= \frac{C}{3}\left[y^2/2\right]_0^2 = \frac{2C}{3} = 1 \Rightarrow C = \frac{3}{2}$$

Answer:

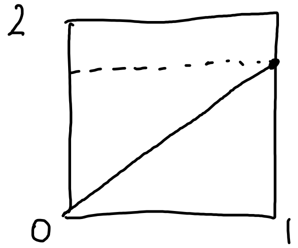
b) Compute $E[XY]$.

$$C\int_0^2\int_0^1 x^3y^2\,dx\,dy = C\int_0^2 y^2/4\,dy = C\left[y^3/12\right]_0^2 = C\frac{8}{12}$$

$$= \frac{\cancel{3}}{\cancel{2}}\,\frac{\cancel{8}^{\cancel{2}}}{\cancel{4}\,\cancel{12}} = 1$$

Answer:

c) Compute $P(X < Y)$.



$$\int_0^1\int_x^2 Cx^2 y\,dy\,dx = \int_0^1 Cx^2\left[\frac{y^2}{2}\right]_x^2 dx$$

$$= \int_0^1 Cx^2\left(2-\frac{x^2}{2}\right)dx = C\left[\frac{2x^3}{3}-\frac{x^5}{10}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{2}{3}-\frac{1}{10}\right] = \frac{3}{2}\,\frac{20-3}{30} = \frac{3\cdot 17}{60} = \frac{17}{20}.$$

Answer:

5. (10 points) Consider the following modification of the birthday problem: There are 50 students in your class. You want to find the average number of pairs of people who have the same birthday. Assume that each person has a birthday uniformly distributed on any of the 365 days of the year (no leap years). Let

$$X_{ij} = \begin{cases} 1 & \text{student } i \text{ and } j \text{ have the same birthday} \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E[X_{ij}]$.

Solution. The chance that two given different students have the same birthday is $\frac{1}{365}$, since there are 365^2 possible pairs of birthdays (x, y) and for 365 of these we have $x = y$, which gives $\frac{365}{365^2} = \frac{1}{365}$.

Answer: $\frac{1}{365}$

(b) Find the expected number of pairs of people who have the same birthday.

Solution. There are $\binom{50}{2}$ pairs of students so by linearity of expectation, the mean of the sum of the X_{ij} across all $i \neq j$ is $\frac{\binom{50}{2}}{365}$. (This does not have to be simplified but if it is, you get $\frac{1225}{365} = \frac{245}{73}$)

Answer: $\frac{\binom{50}{2}}{365}$

6. (20 points) Suppose you flip a fair coin 3 times and record the sequence of heads and tails. Let X be the number of heads in the sequence and Y be the length of the longest consecutive sequence of heads. For example, the sequence HTHHT has 2 consecutive sequences of heads: H and HH. The length of the longest consecutive sequence of heads is 2.

a) Find the joint probability mass function of (X, Y). Write your answers in the table below.

X \ Y	0	1	2	3
0	1/8	0	0	0
1	0	3/8	0	0
2	0	1/8	2/8	0
3	0	0	0	1/8

$P(X=2, Y=2) = P(\{HHT, THH\}) = \frac{2}{8}$

6 pts for everything right

b) Find the marginal probability mass functions $p_X(k)$ of X and $p_Y(k)$ of Y. Write your answers in the table below.

	k = 0	1	2	3
$p_X(k)$	1/8	3/8	3/8	1/8

2 pts

	k = 0	1	2	3
$p_Y(k)$	1/8	4/8	2/8	1/8

2 pts

c) Find Var(X) , Var(Y), and Cov(X,Y).

partial credit

$$E[X^2] = 0\cdot\frac{1}{8} + 1^2\frac{3}{8} + 4\cdot\frac{3}{8} + \frac{9\cdot 1}{8} = \frac{24}{8}$$

$$E[X] = 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8} = \frac{12}{8} = \frac{3}{2}$$

partial credit

$$E[Y^2] = 0\cdot\frac{1}{8} + 1\cdot\frac{4}{8} + 4\cdot\frac{2}{8} + 9/8 = \frac{21}{8}$$

$$E[Y] = 0\cdot\frac{1}{8} + 4/8 + 2\cdot\frac{2}{8} + 3\cdot\frac{1}{8} = 11/8$$

$$\text{Var}(X) = \frac{24}{8} - \frac{9}{4} = \frac{24-18}{8} = \frac{6}{8} = \frac{3}{4}$$

$$\text{Var}(Y) = \frac{21}{8} - \frac{121}{64} = \frac{168-121}{64} = \frac{47}{64}$$

$$\begin{array}{r} 168 \\ 121 \\ \hline 47 \end{array}$$

partial credit

$$E[XY] = \frac{3}{8} + 2\cdot\frac{1}{8} + 4\cdot\frac{1}{4} + 9\cdot\frac{1}{8} = \frac{3+2+8+9}{8} = \frac{22}{8} = \frac{11}{4}$$

$$\text{Cov}(X,Y) = \frac{11}{4} - \frac{3}{2}\cdot\frac{11}{8} = \frac{44-33}{16} = \frac{11}{16}$$

Var(X): $\frac{3}{4}$] 2 pts

Var(Y): $\frac{47}{64}$] 2 pts

Cov(X,Y): $\frac{11}{16}$] 2 pts

d) Find the correlation coefficient of X and Y. You do not need to simplify numerical expressions.

$$\text{Corr} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$$

2 pts.

$$= \frac{11/16}{\sqrt{3/4 \cdot 47/64}} = 0.6550$$

Answer:

e) Are X and Y independent? Justify your answer.

No since

$$P(X=0, Y=1) = \frac{1}{8} \neq P(X=0)P(Y=1)$$

Answer:

7. (10 points) The number of customers coming to your store is modeled by a Poisson random variable with parameter 10, which indicates that on average, 10 people come to your store per day. Let Y the total number of people who come to your store over a period of 1000 days. Assume that the number of customers coming to your store on different days are independent random variables.

(a) Give an exact expression for the probability that at least 10200 people come to your store in 1000 days. *Hint: If* $X_1 \sim \text{Poisson}(\lambda)$ *and* $X_2 \sim \text{Poisson}(\lambda)$, *represent the number of visitors visitors to your store on days 1 and 2, what is the distribution of* $X_1 + X_2$, *the total number of visitors to your store on days 1 and 2? Can you generalize this idea?* You may leave your answer as a sum.

$M_X(t) = e^{10(e^t - 1)}$ (from the formula sheet)

$Y = X_1 + X_2 + \cdots + X_{1000}$ $X_i = $ # of people on day i

$M_Y(t) = \left[e^{10(e^t-1)}\right]^{1000} = e^{10 \cdot 1000 (e^t - 1)}$

$\Rightarrow Y \sim \text{Poisson}(10000)$

$P(Y \geq 10200) = \sum_{k=10200}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$ $\lambda = 10000$

5 points. Partial credit for guessing that Y is Poisson without reasoning (2pts). 1 pt for getting λ correct.

Answer:

1 pt for writing Y as a sum of Poisson.

(b) Find the mean μ_Y and the variance σ_y^2 of Y.

$$\left.\begin{aligned} Var(Y) &= \lambda = 10{,}000 \\ E[Y] &= \lambda = 10{,}000 \end{aligned}\right] \text{ 2 pts}$$

Answer:

(c) Give an estimate for the probability that at least 10200 people come to your store in 1000 days. *Hint: We learned the theorem used here on the last day of class.*

$$P(Y \geq 10200) = \left. P\left(\frac{Y - 10000}{\sqrt{10000}} \geq \frac{200}{\sqrt{10000}} \right) \right] \text{ 2pts}$$

$$\left.\begin{aligned} \approx P\left(Z \geq \frac{2\cancel{00}}{1\cancel{00}}\right) &= 1 - \Phi(2) \\ &= 0.0228 \end{aligned}\right] \text{ 1 pt}$$

Answer:

0.0228

8. (15 points)

Let D be the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. Suppose that the random point (X,Y) is uniformly distributed on D.

(a) Find $E[X]$.

Solution. The triangle is the region bounded above and to the left by $y = x$, below by $y = 0$, and to the right by $x = 1$. This triangle has area $1/2$. Thus, the joint probability density function is give by $f_{X,Y}(x,y) = 2$ for (x,y) in the triangle and $f_{X,Y}(x,y) = 0$ otherwise. Thus, we have

$$E[X] = \int_0^1 \int_0^x 2x \, dy \, dx = \int_0^1 2x^2 \, dx = \frac{2}{3}$$

Answer: $\frac{2}{3}$

(b) Find $E[Y]$.

Solution. This time we have

$$E[Y] = \int_0^1 \int_0^x 2y \, dy \, dx = \int_0^1 x^2 \, dx = \frac{1}{3}$$

Answer: $\frac{1}{3}$

(c) Find $\mathrm{Cov}(X, Y)$.

Solution. Since $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$, we compute

$$E[XY] = \int_0^1 \int_0^x 2xy\, dy\, dx = \int_0^1 x^3\, dx = \frac{1}{4}$$

We have $\frac{1}{4} - \frac{2}{9} = -\frac{1}{36}$.

Answer: $-\frac{1}{36}$

EXTRA SPACE. Use this space if you run out elsewhere. Be sure to label your problems and also include a note on the original page telling the graders to look for your work here.

Table of values for $\Phi(x)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998