MATH 201: Intro to Probability

Midterm 2

March 28, 2024

Name: ______________________________ (Please print clearly)

UR ID: ______________________________

Instructions:

- You have 75 minutes to work on this exam. You are responsible for checking that this exam has all 9 pages. **Please do not remove any pages.**
- Write your final answer in the box provided.
- No calculators, phones, electronic devices, books, or notes are allowed during the exam, except for the provided formula sheet.
- **Show all work and justify all answers.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Numerical or algebraic simplifications of answers are not required, **except when specifically stated otherwise.** However, summations should be evaluated, and expressions such at "$\sum$" and "$\ldots$" should not appear in your final answers.
- Please write your UR ID in the space provided at the top of each page.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

FOR REFERENCE, NO QUESTIONS ON THIS PAGE

For a random variable X,

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

If $X \sim \mathrm{Bin}(n, p)$, then

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, \ldots, n$$

$$E[X] = np, \qquad \mathrm{Var}(X) = np(1-p).$$

If $X \sim \mathrm{Geom}(p)$, then

$$P(X = k) = (1-p)^{k-1} p, \qquad k = 1, 2, 3, \ldots$$

$$E[X] = \frac{1}{p}, \qquad \mathrm{Var}(X) = \frac{1-p}{p}.$$

If $X \sim \mathrm{Poisson}(\lambda)$, then

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

$$E[X] = \lambda, \qquad \mathrm{Var}(X) = \lambda.$$

1. (10 points)

Let X be a random variable with mean $E[X] = 5$ and variance $\mathrm{Var}(X) = 2$.

(a) Compute $E[2x+1]$.

Solution.

$$E[2X+1] = 2E[X] + 1 = 11.$$

(b) Compute $\mathrm{Var}(2X+1)$.

Solution.

$$\mathrm{Var}(2X+1) = 2^2\mathrm{Var}(X) = 8.$$

(c) Compute $E[2X^2+1]$.

Solution.

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 \implies E[X^2] = \mathrm{Var}(X) + (E[X])^2 = 27,$$
$$E[2X^2+1] = 2E[X^2] + 1 = 2 \cdot 27 + 1 = 55.$$

2. (10 points)
Let $X \sim \mathcal{N}(-1.5, 4)$.

(a) Find the probability $P(-2 \leq X < 1)$. You may leave your answer in terms of Φ, where Φ is the cumulative distribution function of a standard normal random variable..

Solution.

Let $Z = \frac{X+1.5}{2}$. Then $Z \sim \mathcal{N}(0,1)$.

$$
\begin{aligned}
P(-2 \leq X < 1) &= P(-0.5 \leq X + 1.5 < 2.5) \\
&= P\left(-0.25 \leq \frac{X+1.5}{2} < 1.25\right) \\
&= P(-0.25 \leq Z < 1.25) \\
&\approx \Phi(1.25) - (1 - \Phi(0.25)) \\
&\approx 0.8944 - (1 - 0.5987) \\
&= 0.4931
\end{aligned}
$$

(b) Find number a and b such that $P(a \leq X < b) = \Phi(3) - \Phi(1)$.

Solution.

$$
\begin{aligned}
\Phi(3) - \Phi(1) &= P(1 \leq Z < 3) \\
&= P\left(1 \leq \frac{X+1.5}{2} < 3\right) \\
&= P(2 \leq X + 1.5 < 6) \\
&= P(0.5 \leq X < 4.5).
\end{aligned}
$$

So, $a = 0.5, b = 4.5$.

3. (10 points)

Suppose that X is a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{4}{x^5}, & x \geq 1, \\ 0, & x < 1. \end{cases}$$

(a) Compute the expectation $E[X]$ of X.

Solution.

$$\begin{aligned} E[X] &= \int_1^\infty x \frac{4}{x^5}\, dx \\ &= 4 \int_1^\infty x^{-4}\, dx \\ &= 4 \Big[\frac{1}{-3x^3} \Big]_1^\infty \\ &= 4/3. \end{aligned}$$

(b) Compute the variance $\mathrm{Var}(X)$ of X.

Solution.

First compute $E[X^2]$:

$$\begin{aligned} E[X^2] &= \int_1^\infty x^2 \frac{4}{x^5}\, dx \\ &= 4 \int_1^\infty \frac{1}{x^3}\, dx \\ &= 4 \Big[\frac{1}{-2x^{-2}} \Big]_1^\infty \\ &= 2. \end{aligned}$$

Therefore

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 2 - (4/3)^2 = 2/9$$

4. (10 points)
Suppose that you have a biased coin; the probability that you flip a tails is $3/5$. Flip this coin 1000 times.

(a) Use the normal distribution to approximate the probability that you flip at least 580 tails. Write your answer in terms of Φ, where Φ is the cumulative distribution function of the standard normal random variable. You do not need to evaluate Φ.

Solution.

Let S be the number of times you flip a tails. Then $S \sim \text{Bin}(1,000, 3/5)$ and so

$$E[S] = 600, \quad \text{Var}(S) = 1000(3/5)(2/5) = 240.$$

So

$$\begin{aligned} P(S \geq 580) &= 1 - P(S < 580) \\ &= 1 - P\left(\frac{S - 600}{\sqrt{240}} < \frac{580 - 600}{\sqrt{240}}\right) \\ &\approx 1 - P\left(Z < \frac{-20}{\sqrt{240}}\right) \\ &= 1 - \Phi\left(\frac{-20}{\sqrt{240}}\right) \\ &= \Phi\left(\frac{20}{\sqrt{240}}\right). \end{aligned}$$

(b) Use the normal distribution to approximate the probability that you flip exactly 600 tails. Write your answer in terms of Φ, where Φ is the cumulative distribution function of the standard normal random variable. You do not need to evaluate Φ.

Solution.

Let S be the number of times you flip a tails. Then $S \sim \text{Bin}(1,000, 3/5)$ and so

$$E[S] = 600, \quad \text{Var}(S) = 1000(3/5)(2/5) = 240.$$

We want to estimate $P(S = 600)$. Using the continuity correction for normal approximation, we get

$$
\begin{aligned}
P(S = 600) &= P(599.5 \le S \le 600.5) \\
&= P\left(\frac{599.5 - 600}{\sqrt{240}} \le \frac{S - 600}{\sqrt{240}} \le \frac{600.5 - 600}{\sqrt{240}}\right) \\
&\approx P\left(\frac{-0.5}{\sqrt{240}} \le Z \le \frac{0.5}{\sqrt{240}}\right) \\
&= \Phi\left(\frac{0.5}{\sqrt{240}}\right) - \Phi\left(\frac{-0.5}{\sqrt{240}}\right) \\
&= \Phi\left(\frac{0.5}{\sqrt{240}}\right) - \left(1 - \Phi\left(\frac{0.5}{\sqrt{240}}\right)\right) \\
&= 2\Phi\left(\frac{0.5}{\sqrt{240}}\right) - 1.
\end{aligned}
$$

5. (10 points)

We are using a poll to determine the percentage of university students that drink coffee. Out of 1000 people, 750 respond that they do drink coffee. Find the 95% confidence interval for the true probability that a randomly chosen student drinks coffee. You may leave your answer in terms of Φ or Φ^{-1}.

Solution:

Our estimated parameter is $\hat{p} = \frac{750}{1000} = .75$. Using the normal approximation, we know that for any $\epsilon > 0$,

$$P(|p - \hat{p}| < \epsilon) \geq 2\Phi(2\epsilon\sqrt{n}) - 1.$$

Thus we want

$$\begin{aligned} .95 &\leq 2\Phi(2\epsilon\sqrt{n}) - = 2\Phi(20\sqrt{10}\epsilon) - 1 \\ \iff .975 &\leq \Phi(20\sqrt{10}\epsilon) \\ \iff \frac{\Phi^{-1}(.975)}{20\sqrt{10}} &\leq \epsilon. \end{aligned}$$

Thus, the 95% confidence interval is

$$(.75 - \frac{\Phi^{-1}(.975)}{20\sqrt{10}}, .75 + \frac{\Phi^{-1}(.975)}{20\sqrt{10}}).$$

6. (10 points)

A manufacturer produces light-bulbs that are packed into boxes of 100. Quality control studies indicate that 0.5% of the light-bulbs produced are defective.

(a) Estimate the probability that a randomly chosen box contains no defective light-bulbs.

Solution.

We use the Poisson approximation. Define X to be the number of defective bulbs in the box. Let $n = 100$ and $p = 0.005$. Then $X \sim \text{Poisson}(\lambda)$ where

$$\lambda = np = 100 \times 0.005 = 0.5.$$

We know the probability mass function of X is given by

$$P(X = k) = \frac{e^{-0.5}(0.5)^k}{k!}.$$

Thus

$$P(X = 0) = \frac{e^{-0.5}(0.5)^0}{0!} = e^{-0.5}.$$

(b) Estimate the probability that a randomly chosen box contains 2 or more defective light-bulbs. (Answer must be explicit, not written in summation notation.)

Solution.

Using the Poisson approximation as in part (a):

$$P(\text{2 or more defectives}) = 1 - P(X = 0) - P(X = 1).$$

Since

$$P(X = 1) = \frac{e^{-0.5}(0.5)^1}{1!} = \frac{e^{-0.5}}{2},$$

$$P(\text{2 or more defectives}) = 1 - e^{-0.5} - \frac{e^{-0.5}}{2} = 1 - \frac{3e^{-0.5}}{2}.$$