MATH 201: Intro to Probability

Midterm 1

February 20, 2024

Name: ________________________________(Please print clearly)

UR ID: ________________________________

Instructions:

- You have 75 minutes to work on this exam. You are responsible for checking that this exam has all 8 pages. **Please do not remove any pages.**
- Write your final answer in the box provided.
- No calculators, phones, electronic devices, books, or notes are allowed during the exam, except for the provided formula sheet.
- **Show all work and justify all answers.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Numerical or algebraic simplifications of answers are not required, **except when specifically stated otherwise.** However, summations should be evaluated, and expressions such at "$\sum$" and "$\ldots$" should not appear in your final answers.
- Please write your UR ID in the space provided at the top of each page.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ________________________________

FOR REFERENCE, NO QUESTIONS ON THIS PAGE

If $X \sim \text{Bin}(n, p)$, then

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, \ldots, n$$

If $X \sim \text{Geom}(p)$, then

$$P(X = k) = (1-p)^{k-1} p, \qquad k = 1, 2, 3, \ldots$$

1. (10 points)

Suppose that in a group of 100 students,

- 15 study math;
- 25 study statistics;
- 30 study computer science.

There are 5 students who double major in math and statistics, 7 who double major in computer science and statistics, and 6 double major in math and computer science. No one studies all three. What is the probability that a randomly selected student does not study math, computer science, or statistics?

Solution:

Let $M = \{$ student studies math$\}$, $S = \{$ student study statistics$\}$, $C = \{$ student studies computer science$\}$. We want to know the probability that a student does not study math, stats, or CS; this is the event $(M \cup S \cup C)^c$. We have

$$P((M \cup S \cup C)^c) = 1 - P(M \cup S \cup C)$$

and we can use inclusion/exclusion to compute $P(M \cup S \cup C)$:

$$\begin{aligned} P(M \cup S \cup C) &= P(M) + P(S) + P(C) - P(MS) - P(MC) - P(SC) + P(MSC) \\ &= \frac{15}{100} + \frac{25}{100} + \frac{30}{100} - \frac{5}{100} - \frac{6}{100} - \frac{7}{100} + 0 \\ &= \frac{52}{100}. \end{aligned}$$

The desired probability is then

$$P((M \cup S \cup C)^c) = 1 - \frac{52}{100} = \frac{48}{100}.$$

2. (10 points)
Let X be a discrete random variable with set of possible values $\{1, 2, 3, ...\}$ and whose probability mass function is given by

$$p_X(k) = \frac{C}{3^k}$$

for k a positive integer, and where C is an unknown constant.

(a) Find the constant C.

Solution:
Using the geometric series formula and the fact that $\sum_k p_X(k) = 1$, we have

$$1 = \sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} \frac{C}{3^k} = C\frac{1/3}{1-1/3} = \frac{C}{2}.$$

So $C = 2$.

(b) Find the probability that X is an even integer.

Solution:

$$\begin{aligned} P\{X \text{ is even}\} &= P\{X = 2\} + P\{X = 4\} + ... \\ &= p_X(2) + p_X(4) + ... + p_X(2k) + ... \\ &= \sum_{k=1}^{\infty} p_X(2k) \\ &= \sum_{k=1}^{\infty} \frac{2}{3^{2k}} \\ &= 2\frac{1/9}{1-1/9} \\ &= \frac{1}{4}. \end{aligned}$$

(c) Given any positive integer N, find the probability the probability that $X \leq N$. (Your answer should be a function of N.)

Solution:
For any positive integer N,

$$P\{X \leq N\} = \sum_{k=1}^{N} p_X(k) = \sum_{k=1}^{N} \frac{2}{3^k} = 2\frac{1/3 - 1/3^{N+1}}{1-1/3} = 1 - \frac{1}{3^N}$$

3. (10 points)
We choose a number from the set $\{10, 11, 12, \ldots, 99\}$ uniformly at random.

(a) Let X be the first digit and Y be the second digit of the chosen number. Show that X and Y are independent random variables.

Solution:
The cardinality of $\{10, 11, 12, \ldots, 99\}$ is 90. Given any $a \in \{1, 2, \ldots, 9\}$ and $b \in \{0, 1, \ldots, 9\}$, we have

$$P(X = a, Y = b) = \frac{1}{90} \qquad \text{and} \qquad P(X = a) = \frac{10}{90},\ P(Y = b) = \frac{9}{90}.$$

Thus we have $P(X = a, Y = b) = P(X = a) \cdot P(Y = b)$, so X and Y are independent.

(b) Let X be the first digit of the chosen number and Z the sum of the two digits. Show that X and Z are **not** independent random variables.

Solution:
If we consider $X = 1$ and $Z = 9$, from part (a) we have

$$P(X = 1, Z = 9) = P(X = 1, Y = 8) = \frac{1}{90}$$

We also know that $P(X = 1) = \frac{1}{9}$. However,

$$P(Z = 9) = P(X = 1, Y = 8) + P(X = 2, Y = 7) + \cdots + P(X = 8, Y = 1) = \frac{8}{90}.$$

That is

$$P(X = 1, Z = 9) \neq P(X = 1) \cdot P(Z = 9)$$

Hence, X and Z are not independent.

4. (10 points)

Suppose we have 3 boxes of chalk: **box 1** contains 2 pieces of white chalk and 3 pieces of yellow chalk, **box 2** contains 4 pieces of yellow chalk, 1 piece of white chalk, and 1 piece of red chalk, and **box 3** contains 3 pieces of red chalk and 3 pieces of white chalk. Choose a box uniformly at random.

(a) Suppose we draw a piece of chalk at random from our chosen box, and it is yellow. Given this information, what is the conditional probability that we chose box 1?

Solution:

Let $B1$ be the event that we chose box 1, $B2$ the event we chose box 2, and $B3$ the event we chose box 3. Let Y be the event that the chosen chalk is yellow. Then

$$\begin{aligned} P(B1|Y) &= \frac{P(Y|B1)P(B1)}{P(Y|B1)P(B1) + P(Y|B2)P(B2) + P(Y|B3)P(B3)} \\ &= \frac{\frac{3}{5}\cdot\frac{1}{3}}{\frac{3}{5}\cdot\frac{1}{3} + \frac{4}{6}\cdot\frac{1}{3} + 0\cdot\frac{1}{3}} = \frac{\frac{1}{5}}{\frac{1}{5}+\frac{2}{9}} \\ &= \frac{9}{19}. \end{aligned}$$

(b) Suppose instead that we draw two pieces without replacement. What is the probability that the two pieces of chalk we draw are the same color?

Solution:

Let S be the event that the two pieces of chalk are the same color. Then

$$S = YY \cup RR \cup WW.$$

We compute the probability for each box:

$$P(S|B1) = P(YY|B1) + P(WW|B1) + P(RR|B1) = \frac{\binom{3}{2}}{\binom{5}{2}} + \frac{\binom{2}{2}}{\binom{5}{2}} + 0 = \frac{2}{5}$$

$$P(S|B2) = P(YY|B2) + P(WW|B2) + P(RR|B2) = \frac{\binom{4}{2}}{\binom{6}{2}} + 0 + 0 = \frac{2}{5}$$

$$P(S|B3) = P(YY|B3) + P(WW|B3) + P(RR|B3) = 0 + \frac{\binom{3}{2}}{\binom{6}{2}} + \frac{\binom{3}{2}}{\binom{6}{2}} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}.$$

Therefore

$$P(S) = P(S|B1)P(B1) + P(S|B2)P(B2) + P(S|B3)P(B3) = 3\cdot\frac{2}{5}\cdot\frac{1}{3} = \frac{2}{5}.$$

5. (10 points) Major universities claim that 80% of the senior athletes graduate in any particular year. Senior athletes attending major universities are randomly selected to form an ordered list, and we record whether each graduated or not.

(a) What is the probability that the first senior athlete to graduate from the selected group was the 5^{th} selected?

Solution:
Let X be the number of selected senior athletes until the first senior athlete to graduate appear. We have $X \sim Geom\,(4/5)$.

$$P\,(X = 5) = \left(1 - \frac{4}{5}\right)^4 \cdot \frac{4}{5} = \frac{4}{5^5}.$$

(b) What is the probability that the first senior athlete to graduate from the selected group was within the first 3 selected?

Solution:

$$\begin{aligned} P\,(X \leq 3) &= P\,(X = 1) + P\,(X = 2) + P\,(X = 3) \\ &= \left(1 - \frac{4}{5}\right)^0 \cdot \frac{4}{5} + \left(1 - \frac{4}{5}\right) \cdot \frac{4}{5} + \left(1 - \frac{4}{5}\right)^2 \cdot \frac{4}{5} \\ &= \frac{124}{125}. \end{aligned}$$

6. (10 points)
Suppose you need to complete a multiple choice exam. There are 20 questions, and each has 4 possible answers.

(a) Suppose you choose your answers at random. What is the probability that you score at least 90%?

Solution:
Let X be the number of questions you answer correctly–then $X \sim \text{Bin}\left(20, \frac{1}{4}\right)$.

$$\begin{aligned} P(X \geq 18) &= P(X = 18) + P(X = 19) + P(X = 20) \\ &= \binom{20}{18}\left(\frac{1}{4}\right)^{18}\left(\frac{3}{4}\right)^{2} + \binom{20}{19}\left(\frac{1}{4}\right)^{19}\left(\frac{3}{4}\right) + \binom{20}{20}\left(\frac{1}{4}\right)^{20}. \end{aligned}$$

(b) Suppose instead that you know the correct answer with 50% probability, and if you don't know the answer, you will choose at random. What is the probability that you score at least 90%?

Solution: Let K be the event that you know the answer to a given question. Then the probability of the event C of answering a particular question correctly is

$$P(C) = P(C|K)P(K) + P(C|K^C)P(K^C) = 1 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{5}{8}.$$

Again, let X be the number of questions you answer correctly. Now, $X \sim \text{Bin}\left(20, \frac{5}{8}\right)$, so

$$\begin{aligned} P(X \geq 18) &= P(X = 18) + P(X = 19) + P(X = 20) \\ &= \binom{20}{18}\left(\frac{5}{8}\right)^{18}\left(\frac{3}{8}\right)^{2} + \binom{20}{19}\left(\frac{5}{8}\right)^{19}\left(\frac{3}{8}\right) + \binom{20}{20}\left(\frac{5}{8}\right)^{20}. \end{aligned}$$