MATH 201: Written Homework 9
Due Friday, 6/21 by 1pm EDT

(P1) Let $X \sim \text{Binomial}(n, p)$.

(a) Compute the moment generating function of X.

(b) Compute $E[X^3]$.

(P2) Let $Z_1, \ldots, Z_n$ be independent normal random variables with mean 0 and variance 1 and define

$$Y = Z_1^2 + \cdots + Z_n^2.$$

(a) Find the moment generating function of Z_1^2. Hint: use the definition $M_{Z_1^2}(t) = E[e^{tZ_1^2}]$ and be careful integrating, as $M_{Z_1^2}(t)$ is only finite for $t < 1/2$.

(b) Compute the mean and variance of Y.

(c) Find the moment generating function $M_Y(t)$ of Y.

(d) Compute $E[Y^3]$.

(P3) For a positive integer n and $\lambda > 0$, a variable $Y \sim \text{Gamma}(n, \lambda)$ is *Gamma distributed* if it has density (see pg. 168 of our text)

$$f_Y(t) = \begin{cases} \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $X_1, \ldots, X_n$ be independent exponentially distributed random variables, each with the same parameter λ. In this problem you will show that $X_1 + \ldots + X_n \sim \text{Gamma}(n, \lambda)$.

(a) Recall at the end of our 8.3 notes, we discussed that $X_1 + X_2 \sim \text{Gamma}(2, \lambda)$ using convolution. Use convolution to show that $X_1 + X_2 + X_3 \sim \text{Gamma}(3, \lambda)$.

(b) More generally, use convolution to show that if k is a positive integer, $X \sim Exp(\lambda)$, and $Y \sim \text{Gamma}(k, \lambda)$, then $X + Y \sim \text{Gamma}(k + 1, \lambda)$. Hint: you will need to compute the integral $\int_0^t x^{k-1} e^{-\lambda x} e^{-\lambda(t-x)}\, dx$.

(c) Observe that $\text{Gamma}(1, \lambda)$ is precisely $Exp(\lambda)$ simply by definition. Briefly explain how this and part (b) allow us to conclude that $X_1 + \ldots + X_n \sim \text{Gamma}(n, \lambda)$ (i.e. what proof technique is used?).