Math 201

Final Exam ANSWERS

December 18, 2015

Table of values for $\Phi(x)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

1. (12 points)

You have 8 friends. You will invite 5 of them to a party.

(a) How many choices are there if 2 of the friends are fighting and will not attend together?

(b) How many choices are there if 2 of the friends will only attend together?

Answer:

(a) $\binom{6}{5} + \binom{2}{1}\binom{6}{4}$

(b) $\binom{6}{5} + \binom{6}{3}$

2. (12 points)

The entire output of a factory is produced on three machines. The three machines account for 10%, 40%, and 50% of the output, respectively. For the first machine, the percentage of items produced that are defective is 15%. For the second machine, the percentage of items produced that are defective is 5%. For the third machine, the percentage of items produced that are defective is 4%.

If an item is chosen at random from the total output and is found to be defective, what is the probability that it was produced by the first machine?

Answer:

Let $A_i = \{\text{item made by } i^{\text{th}} \text{ machine}\}$ and $B = \{\text{item defective}\}$. We have

$$P(A_1) = 0.1, \qquad P(A_2) = 0.4, \qquad P(A_3) = 0.5,$$

and

$$P(B|A_1) = 0.15, \qquad P(B|A_2) = 0.05, \qquad P(B|A_3) = 0.04.$$

Then

$$\begin{aligned}
P(A_1|B) &= \frac{P(B|A_1)P(A_1)}{P(B)} \\
&= \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)} \\
&= \frac{(0.15)(0.1)}{(0.15)(0.1) + (0.05)(0.4) + (0.04)(0.5)} \\
&= \frac{0.015}{0.055} = \frac{3}{11}
\end{aligned}$$

3. (12 points)

Consider two rolls of a fair six-sided die and the following events:

$$A = \{\text{1st roll is 1, 2, or 3}\}$$
$$B = \{\text{1st roll is 3, 4, or 5}\}$$
$$C = \{\text{the sum of the two rolls is 9}\}$$

Are the events A, B, C independent? Justify your answer with a calculation.

Answer:

No, they are dependent.

Any **one** of the following three calculations shows that the events are dependent.

$$P(A \cap B) = \frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)$$
$$P(A \cap C) = \frac{1}{36} \neq \frac{1}{2} \cdot \frac{4}{36} = P(A)P(C)$$
$$P(B \cap C) = \frac{1}{12} \neq \frac{1}{2} \cdot \frac{4}{36} = P(B)P(C)$$

Other justifying calculations are possible. For example,

$$P(C) = \frac{4}{36} \neq \frac{2}{36} = P(C|A).$$

Remark: We have

$$P(A \cap B \cap C) = \frac{1}{36} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4}{36} = P(A)P(B)P(C)$$

But this equation alone does not mean the events are independent.

4. (12 points)

At a school of 10,000 students, two language classes are offered: Norwegian and Mongolian. 5,000 students are taking at least one language class. 3,000 are taking Norwegian and 4,000 are taking Mongolian. If a student is taking Norwegian, what is the probability the student is also taking Mongolian?

Answer:

Let $A = \{\text{student taking Norwegian}\}$ and $B = \{\text{student taking Mongolian}\}$. We have $P(A) = 0.3$, $P(B) = 0.4$, and $P(A \cup B) = 0.5$. The desired probability is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

By inclusion-exclusion,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

Therefore

$$P(B|A) = \frac{P(A) + P(B) - P(A \cup B)}{P(A)} = \frac{0.3 + 0.4 - 0.5}{0.3} = \frac{2}{3}$$

5. (12 points) Consider an infinite sequence of coins: $\text{coin}_1, \text{coin}_2, \text{coin}_3, \ldots$ The probability of heads for coin_i is p_i.

Flip each coin in order, starting with coin_1. Let X be the number of flips until the first heads. For example, for the sequence of flips TTHHTHHH..., we have $X = 3$.

Fill in the boxes below. Write your answers as simply as possible. An answer involving (for example) an infinite summation will earn only partial credit.

(a) The probability mass function of X is

$p_X(k) =$ [ ] for $k = 1, 2, 3, \ldots$

(b) The probability that $X > k$ is

$P(X > k) =$ [ ] for $k = 1, 2, 3, \ldots$

Answer:

(a)

$$p_X(k) = (1 - p_1) \cdot \cdots \cdot (1 - p_{k-1}) p_k \qquad \text{for } k = 1, 2, 3, \ldots$$

(b)

$$P(X > k) = (1 - p_1) \cdot \cdots \cdot (1 - p_k) \qquad \text{for } k = 1, 2, 3, \ldots$$

6. (15 points)

Let X and Y be random variables with joint density function

$$f_{X,Y}(x, y) = \begin{cases} \dfrac{xy}{9}, & x \in [0, 3], y \in [0, 2], \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $E[XY]$.

(b) Find $P(X < Y)$.

Answer:

(a) We have

$$E[XY] = \int_0^3 \int_0^2 xy \frac{xy}{9} dy dx = \int_0^3 \int_0^2 \frac{x^2 y^2}{9} dy dx = \int_0^3 \frac{x^2 y^3}{27} |_{y=0}^2 dx$$
$$= \int_0^3 \frac{8x^2}{27} dx = \frac{8x^3}{81} |_{x=0}^3 = \frac{8}{3}.$$

(b) We have

$$P(X < Y) = \int_0^2 \int_0^y \frac{xy}{9} dx dy = \int_0^2 \int_0^y \frac{x^2 y}{18} |_{x=0}^y dy$$
$$= \int_0^2 \frac{y^3}{18} dy = \frac{y^4}{4 \cdot 18} |_{y=0}^2 = \frac{16}{4 \cdot 18} = \frac{2}{9}.$$

7. (15 points) Let X and Y be independent random variables with moment generating functions $M_X(t)$ and $M_Y(t)$. Suppose we know that $M_X(t) M_Y(t) = e^{t^2}$.

(a) What is the distribution of $X + Y$?

(b) What is $E[(X + Y)^3]$?

Answer:

(a) Since X and Y are independent, we have $M_{X+Y}(t) = M_X(t) M_Y(t) = e^{t^2}$. Since the moment generating function of a $N(\mu, \sigma^2)$ normal random variable is $e^{\mu t + \sigma^2 t^2/2}$, we see that we get e^{t^2} when $\mu = 0$ and $\sigma^2 = 2$, so e^{t^2} is the moment generating function of a $N(0, 2)$ random variable. Since a finite moment generating function determines the distribution of a random variable, we have that $X + Y$ must have distribution $N(0, 2)$.

(b) To compute $E[(X + Y)^3]$ we can differentiate its moment generating function 3 times and evaluate at 0. We have

$$\frac{d^3}{dt^3} e^{t^2} = \frac{d^2}{dt^2} 2te^{t^2} = \frac{d}{dt}(2e^{t^2} + 4t^2 e^{t^2}) = 4te^{t^2} + 8te^{t^2} + 8t^3 e^{t^2}.$$

Setting $t = 0$ we get $E[(X + Y)^3] = 0$.

8. (15 points)

You plan to organize a party for the freshmen in the university and would like to have a good estimate of the number of students that will come. For this purpose you conduct a survey of random freshmen asking them whether they plan to come or not. If there are 1000 freshmen, at least how many students do you need to ask to be able to claim that your guess for the number of students that will come will be within 50 of the actual number with probability at least 80%?

Answer:

Let p be the portion of the students that will come to the party. This means $1000p$ students will come. If we ask n students and portion p' of them say they will come, our guess for the number of students who will come will be $1000p'$. Thus, we want to find n such that

$$P(|1000p - 1000p'| < 50) > 0.95.$$

I.e.

$$P(|p - p'| < \frac{1}{20}) > 0.80.$$

Since we have that

$$P(|p - p'| < \varepsilon) \geq 2\Phi(2\varepsilon\sqrt{n}) - 1,$$

n should be such that

$$2\Phi(2\frac{1}{20}\sqrt{n}) - 1 \geq 0.80.$$

I.e. $\Phi(2\frac{1}{20}\sqrt{n}) \geq 0.90$. Consulting the table for Φ, we see that this implies $\sqrt{n}/10 \geq 1.29$ so $n \geq 12.9^2$.

9. (15 points) In a sequence of n flips of a fair coin let X be the number of times it happens that two consecutive flips result in different outcomes. Compute $E[X]$.

To clarify X, consider the following example. If $n = 10$ and the sequence of flips ends up being $HHTHHTTHHT$, then $X = 5$, since the following consecutive flips result in different outcomes: 2nd and 3rd, 3rd and 4th, 5th and 6th, 7th and 8th, 9th and 10th.

Explain your answer.

Answer:

Introduce the indicator random variables $I_1, \ldots, I_{n-1}$ defined as follows: $I_k = 1$ if the k'th and $k+1$'th flips are different and $I_k = 0$ otherwise. We have $X = I_1 + \cdots + I_{n-1}$, so by linearity of expectation $E[X] = E[I_1] + \cdots + E[I_{n-1}]$. Since I_k is an indicator,

$$E[I_k] = P(I_k = 1) = P(\text{the } k\text{'th flip is different from the } k+1\text{'th flip}) = \frac{1}{2},$$

hence

$$E[X] = \frac{1}{2} + \cdots + \frac{1}{2} = \frac{n-1}{2}.$$

Alternate solution:

For $i = 1, 2, \ldots, n$ let F_i be the outcome of flip i and for $j = 1, \ldots, n-1$ let B_i be the event that $F_i = F_{i+1}$. Let's check that the events $B_1, \ldots, B_{n-1}$ are independent. Consider $P(B_1^{c_1} \cdots B_{n-1}^{c_{n-1}})$, where $B_i^{c_i}$ is either B_i or its complement. If $B_1^{c_1} \cdots B_{n-1}^{c_{n-1}}$ is true, then for each neighboring pair of flips we know whether they are the same or not, which means there are exactly two possible outcomes. Thus $P(B_1^{c_1} \cdots B_{n-1}^{c_{n-1}}) = 2/2^n = 1/2^{n-1}$. Since $P(B_i) = P(B_i^c) = 1/2$, we see that $P(B_1^{c_1} \cdots B_{n-1}^{c_{n-1}}) = P(B_1^{c_1}) \cdots P(B_{n-1}^{c_{n-1}})$, which implies the B_i's are independent. Notice that X counts the number of B_i's which are true, hence $X \sim Bin(n-1, 1/2)$, so $E[X] = (n-1)/2$.

10. (14 points) Let $X_1, ..., X_4$ be i.i.d random variables which are distributed Bernoulli with parameter $\frac{1}{3}$. Find $Cov(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$.

Answer:

From independence we have $Cov(X_i, X_j) = 0$ when $i \neq j$ hence

$$Cov(X_1 + X_2 + X_3, X_2 + X_3 + X_4) = Cov(X_2, X_2) + Cov(X_3, X_3) = 2Var(X_1) = \left(\frac{2}{3}\right)^2.$$

11. (15 points) Customers arrive to the store according to a Poisson process with intensity $\lambda = 1$ customers per hour. Let $N[s, t]$ be the number of arrivals in the time interval $[s, t]$. Find the probability $P(N[0, 2] = 2, N[1, 3] = 1)$.

Answer:

$$\begin{aligned}&P(N[0,2] = 2, N[1,3] = 1)\\ &= P(N[0,1] = 1, N[1,2] = 1, N[2,3] = 0) + P(N[0,1] = 2, N[1,2] = 0, N[2,3] = 1).\end{aligned}$$

Therefore

$$P(N[0,2] = 2, N[1,3] = 1) = \left(e^{-1}\right)^3 + \frac{e^{-1}}{2} e^{-1} e^{-1} = \frac{3e^{-3}}{2}.$$

12. (15 points) Let X be the amount of money earned by a food truck on Main Street on a day. Assume that from past experience, the owner of the cart knows that $E[X] = 4,000$, and $Var(X) = 3,000$. Use Chebyshev's inequality to give an upper bound for the probability that the cart will earn at least $7,000$ tomorrow.

Answer:

$$P(X \geq 7000) \leq P(|X - 4000| \geq 3000) \leq \frac{Var(X)}{3000^2} \leq \frac{1}{3000}.$$

13. (15 points) Let X, Y have the joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-y-x^2/2} & \text{if } -\infty < x < \infty,\ 0 < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Find $E(X|Y)$.

Answer:

Clearly $f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot e^{-y}$ when $-\infty < x < \infty$, $0 < y < \infty$, hence X and Y are independent. We get that $f_{X|Y=y}(x) = f_X(x)$ and therefore $E(X|Y = y) = E(X)$. We conclude that $E(X|Y) = E(X) = 0$, since $X \sim N(0,1)$.

Alternate solution:

We have $f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. For $y > 0$ we have

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y-x^2/2} dx = e^{-y} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = e^{-y},$$

where the last equality holds since the integrand is the density of a standard Gaussian. Thus,

$$f_{X|Y=y} = \frac{\frac{1}{\sqrt{2\pi}} e^{-y-x^2/2}}{e^{-y}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

which is the density of a standard Gaussian. Thus $X|Y = y$ is a standard Gaussian, which implies $E[X|Y = y] = 0$ for every $y > 0$. Thus $E[X|Y] = 0$.

14. (21 points)

In each of the following questions identify the described random variable as either binomial, Poisson, geometric, negative binomial, exponential, gamma or hypergeometric and describe a reasonable numerical value for all its parameters. Please use the conventional notation for the parameters i.e., p, λ, n etc. In some cases more than one random variable type might fit, but make sure that you select a choice where you can describe numerical values for all the parameters. No explanation is necessary for this problem. Only give your answers.

(a) A couple has 5 children. X_1 is the number of female children. What type of random variable is X_1 and what are its parameters?

(b) Two dice are rolled over and over again until a sum of 12 occurs. Let X_2 be the number of trials required. What type of random variable is X_2 and what are its parameters?

(c) 100 students are split equally in party affiliation: 50 are Republicans, 50 are Democrats. You select 10 of these students at random to interview. Let X_3 be the number of Republicans that you interview. What type of random variable is X_3 and what are its parameters?

(d) Deaths in the village of Albion occur at an average rate of 5 per year. Let X_4 be the number of deaths in Albion over a decade. What type of random variable is X_4 and what are its parameters?

(e) Someone just died at the village of Albion (see previous part). Let $X_5 \in \mathbb{R}$ be the amount of time, measured in years, the village will go without another death. What type of random variable is X_5 and what are its parameters?

(f) Two barrels (a blue barrel and a red barrel) contain an unlimited amount of fish. At each step you choose one of the barrels with equal likelihood and reach in and take out a fish from it. You stop when you take the 6th fish out of the blue barrel. Let X_6 be the total number of fish you have taken out of both barrels at this point. What type of random variable is X_6 and what are its parameters?

(g) You have 50 candies to give out to trick-or-treaters on Halloween. Suppose you get on average 5 trick-or-treaters per 10 minutes. Let $X_7 \in \mathbb{R}$ be the time, measured in minutes, it takes for you to run out of candy. What type of random variable is X_7 and what are its parameters?

Answer:

(a) Binomial, $n = 5$, $p = 1/2$.

(b) Geometric, $p = 1/36$.

(c) Hypergeometric $N = 100$, $n = 10$, $m = 50$.

(d) Poisson, $\lambda = 5 \cdot 10 = 50$.

(e) Exponential, $\lambda = 5$.

(f) Negative binomial, $k = 6$, $p = 1/2$.

(g) Gamma, $\lambda = 5/10 = 0.5$, $k = 50$.