Math 172: Honors Calculus II

Midterm II

March 27th, 2024

NAME (please print legibly): SOLUTIONS

Your University ID Number: ______

Your University email ______

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______

- **The use of calculators, cell phones, iPods, and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 7 pages.**

QUESTION	VALUE	SCORE
1	10	
2	5	
3	8	
4	10	
5	10	
6	7	
7	10	
TOTAL	60	

1. (10 points) Evaluate the following indefinite integrals.

(a) $\int x \sin x \cos x \, dx = I$

$\underline{\text{IBP}}$: $u = x \qquad v = \frac{1}{2}\sin^2 x$

$du = dx \qquad dv = \sin x \cos x \, dx$

$$I = \frac{1}{2}x\sin^2 x - \frac{1}{2}\int \sin^2 x \, dx = \frac{1}{2}x\sin^2 x - \frac{1}{2}\int \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}x\sin^2 x - \frac{1}{4}x + \frac{1}{8}\sin 2x + C$$

$\underline{\text{Check}}$: $\frac{d}{dx}\left(\frac{1}{2}x\sin^2 x - \frac{1}{4}x + \frac{1}{8}\sin 2x\right) = \cancel{\frac{1}{2}\sin^2 x} + x\sin x\cos x \underbrace{- \frac{1}{4} + \frac{1}{4}\cos 2x}_{\cancel{-\frac{1}{2}\sin^2 x}}$ ✓

$\underline{\text{note}}$: could also use $\sin 2x = 2\sin x \cos x$

(b) For $x > 1$, $\int \frac{\sqrt{x}}{1+x} \, dx = I$

Let $u = \sqrt{x}$ so $u^2 = x$ so $dx = 2u\,du$.

Then $I = \int \frac{u}{1+u^2} 2u\,du = \int \frac{2u^2}{1+u^2} du = 2\int \frac{u^2+1-1}{u^2+1} du$

$$= 2u - 2\arctan u + C$$

$$= 2\sqrt{x} - 2\arctan\sqrt{x} + C$$

$\underline{\text{Check}}$: $\frac{d}{dx}\left(2\sqrt{x} - 2\arctan\sqrt{x}\right) = \cancel{2}\cdot\cancel{\tfrac{1}{2}}x^{-1/2} - \cancel{2}\frac{1}{1+(\sqrt{x})^2}\cdot\cancel{\tfrac{1}{2}}x^{-1/2} = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}(1+x)}$

$$= \frac{(1+x)-1}{\sqrt{x}(1+x)} = \frac{\sqrt{x}}{1+x} \checkmark$$

2. (5 points) Evaluate the following definite integral.

$$\int_0^1 \frac{1}{e^x + e^{-x}}\, dx = I$$

Let $u = e^x$ so $x = \log u$ so $dx = \frac{du}{u}$, and $x = 0 \to u = 1$, $x = 1 \to u = e$

Then $I = \int_1^e \frac{1}{u + u^{-1}} \cdot \frac{du}{u} = \int_1^e \frac{1}{u^2+1}\, du = \arctan(u)\Big|_1^e = \arctan(e) - \arctan(1) + C$

$= \arctan(e) - \pi/4 + C$

3. (8 points) Let $f(x) = \int_1^x \log t\, dt$. Write down $P_{3,1}(x)$, the degree 3 Taylor polynomial for f centered at 1.

$f(x) = \int_1^x \log t\, dt$ $\qquad$ $f(1) = \int_1^1 \log t\, dt = 0$

$f'(x) = \log x$ (FTC1) $\qquad$ $f'(1) = 0$

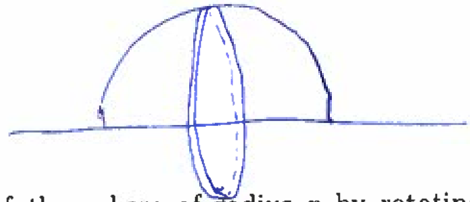
$f''(x) = \frac{1}{x}$ $\qquad$ $f''(1) = 1$

$f^{(3)}(x) = -\frac{1}{x^2}$ $\qquad$ $f^{(3)}(1) = -1$

$P_{3,1}(x) = 0 + 0(x-1) + \frac{1}{2!}(x-1)^2 - \frac{1}{3!}(x-1)^3$

$= \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$

4. (10 points)

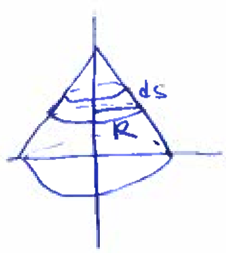


(a) Find the volume of the sphere of radius r by rotating the area under the graph of $y = \sqrt{r^2 - x^2}$ (from $x = -r$ to $x = r$) around the x-axis.

discs: $$\text{Vol} = \int_{-r}^{r} \pi R^2 dx = \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_{-r}^{r} (r^2 - x^2)\,dx = \pi\left(r^2 x - \tfrac{1}{3}x^3\right)\Big|_{-r}^{r}$$

$$= \pi\left(r^2(2r) - \tfrac{1}{3}(2r^3)\right) = 2\pi r^3 - \tfrac{2}{3}\pi r^3 = \tfrac{4}{3}\pi r^3 \checkmark$$

(b) Find the surface area of the cone of radius r and height h by rotating the graph of $y = -\frac{h}{r}x + h$ from $y = 0$ to $y = h$ about the y-axis.



$$SA = \int_0^h 2\pi R\,ds = 2\pi \int_0^h x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = 2\pi \int_0^h -\frac{r}{h}(y-h)\sqrt{1 + \left(\frac{r}{h}\right)^2}\,dy$$

note: $x = -\frac{r}{h}(y-h)$

$\frac{dx}{dy} = -\frac{r}{h}$

$$= \frac{2\pi r}{h}\sqrt{1 + \left(\frac{r}{h}\right)^2} \int_0^h (h - y)\,dy = \frac{2\pi r}{h}\sqrt{1 + \frac{r^2}{h^2}}\left[hy - \tfrac{1}{2}y^2\right]_0^h$$

$$= \frac{\cancel{2}\pi r}{\cancel{h}}\sqrt{1 + \frac{r^2}{h^2}}\left[\tfrac{1}{\cancel{2}}h^{\cancel{2}}\right] = \pi r h\sqrt{1 + \frac{r^2}{h^2}} = \pi r\sqrt{h^2 + r^2} \checkmark$$

You might find the Lagrange form of the Remainder in Taylor's Theorem helpful for the next two problems. Here it is:
Let $R_{n,a}(x)$ denote the n-th remainder term for f, as usual. If $f(x)$ is $n+1$ times differentiable at a, then there is some $t \in [a, x]$ such that

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}.$$

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5. (10 points) Let $f(x) = \cos x$.

(a) Write down $P_{n,0}(x)$ explicitly.

If $n = 2k$ is even $P_{n,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!}$

If $n = 2k+1$ is odd $P_{n,0}(x) = P_{n-1,0}(x)$

(b) Show that $|R_{2n,0}(x)| \le \dfrac{|x|^{2n+2}}{(2n+2)!}$

From (a), $P_{2n+1,0}(x) = P_{2n,0}(x)$ so $R_{2n,0}(x) = R_{2n+1,0}(x)$

By Taylor's thm, $|R_{2n,0}(x)| = |R_{2n+1,0}(x)| = \left| \frac{\cos^{(2n+2)}(t)}{(2n+2)!} (x)^{2n+2} \right|$

for some $t \in [0, x]$.

Since $|\cos^{(2n+2)}(t)| \le 1 \ \forall t$, we get

$$|R_{2n,0}(x)| \le \frac{|x|^{2n+2}}{(2n+2)!}$$

note: Taylor's thm applied to $R_{2n,0}(x)$ gives $R_{2n,0}(x) = \frac{\cos^{(2n+1)}(t)}{(2n+1)!} x^{2n+1}$

NOT $2n+2$

6. **(7 points)** Show that if one uses

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

to approximate e^x on the interval $0 \le x \le \frac{1}{2}$, the error will be $\le .01$.

$1 + x + \frac{x^2}{2} + \frac{x^3}{6} = P_{3,0}(x)$ for e^x

so the error will be $R_{3,0}(x)$, which,

by Taylor's thm is

$|R_{3,0}(x)| = \left|\frac{e^t}{4!} x^4\right|$ for some $t \in [0, x]$ since $\frac{d^n}{dt^n} e^t = e^t$ $\forall n$

e^x is an increasing function and we showed $e < 4$.

Since $t \le x \le \frac{1}{2}$, $e^t \le e^x \le e^{1/2} < 4^{1/2} = 2$

So $|R_{3,0}(x)| < \left|\frac{2}{4!}\left(\frac{1}{2}\right)^4\right| = \frac{1}{4! \cdot 2^3} \le \frac{1}{12 \cdot 16} \le \frac{1}{10 \cdot 10} = 0.01$

7. **(10 points)**

(a) Prove that $\int_1^\infty x^{-x}\,dx$ converges.

Sol'n 1: First note $x^{-x} = \frac{1}{x^x} = \frac{1}{e^{x\log x}} \leq \frac{1}{e^x}$ for $x \geq e$ (so $\log x \geq 1$)

We know $\int_e^\infty e^{-x}$ converges (or you can check $\int_e^\infty e^{-x}dx = \lim_{N\to\infty} -e^{-x}\Big|_e^N = \lim_{N\to\infty} -e^{-N}+e^{-e} = e^{-e}$ conv)

and $\int_1^e x^{-x}dx$ is a constant C since x^{-x} is continuous, hence integrable

so $\int_1^\infty x^{-x}dx = \int_1^e x^{-x}dx + \int_e^\infty x^{-x}dx \leq C + \int_e^\infty e^{-x}dx$ conv so $\int_1^\infty x^{-x}dx$ conv by comparison test

Sol'n 2: first note $x^{-x} = \frac{1}{x^x} = \frac{1}{e^{x\log x}} \leq \frac{1}{e^{2\log x}} = \frac{1}{x^2}$ for $x \geq 2$

So $\int_1^\infty x^{-x}dx = \int_1^2 x^{-x}dx + \int_2^\infty x^{-x}dx \leq \underbrace{\int_1^2 x^{-x}dx}_{\text{const.}} + \underbrace{\int_2^\infty \frac{1}{x^2}dx}_{\text{conv.}}$ so $\int_1^\infty x^{-x}dx$ conv by comparison test

(b) Determine (with an argument) whether the following integral converges or diverges

$$\int_{10}^\infty \frac{1}{x\log x \log\log x}\,dx.$$

note: $\frac{d}{dx}\log\log\log x = \frac{1}{\log\log x}\cdot\frac{1}{\log x}\cdot\frac{1}{x} = \frac{1}{x\log x\log\log x}$

so $\int_{10}^\infty \frac{1}{x\log x\log\log x}dx = \lim_{N\to\infty} \log\log\log x\Big]_{10}^N = \lim_{N\to\infty} \log\log\log N - \log\log\log 10$

$\log 10 > 1$ so $\log\log 10 > 0$ so $\log\log\log 10$ is defined (i.e. just a number)

$\log y \to \infty$ as $y \to \infty$ so $\log N \to \infty$, so $\log\log N \to \infty$, so

$\log\log\log N \to \infty$ as $N \to \infty$

So the integral DIV.