Math 171: Honors Calculus II

Midterm I

February 21st, 2024

NAME (please print legibly): ______________________________
Your University ID Number: ______________________________
Your University email ______________________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

- **The use of calculators, cell phones, iPods, and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 7 pages.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	60	

1. (10 points) Suppose that $f(x)$ is integrable and $f(x) \geq 0$ on $[a, b]$. Prove that $f^2(x)$ (i.e., the function $(f(x))^2$) is integrable on $[a, b]$.

First, f integrable $\implies$ f bounded $\implies$ $\exists M > 0$ such that $|f| \leq M$ on $[a, b]$.

Second, note that $M_i(f^2) \leq (M_i(f))^2$ since $M_i(f)^2$ is an upper bound for f^2, and, similarly, $m_i(f^2) \geq (M_i(f))^2$ since $m_i(f)^2$ is a lower bound for f^2.

Let $\epsilon > 0$.

Because f integrable, $\exists$ partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \frac{\epsilon}{2M}$.

Then we compute

$$U(f^2, P) - L(f^2, P) = \sum_{i=1}^{n} (M_i(f^2) - m_i(f^2))(t_i - t_{i-1})$$

$$\leq \sum_{i=1}^{n} (M_i(f)^2 - m_i(f)^2)(t_i - t_{i-1})$$

$$= \sum_{i=1}^{n} (M_i(f) + m_i(f))(M_i(f) - m_i(f))(t_i - t_{i-1})$$

$$\leq 2M \sum_{i=1}^{n} ((M_i(f) - m_i(f))(t_i - t_{i-1})$$

$$\leq 2M[U(f, P) - L(f, P)] < \epsilon$$

Since $\epsilon > 0$ was arbitrary and we showed $\exists$ partition P of $[a, b]$ such that $U(f^2, P) - L(f^2, P) < \epsilon$, we conclude f^2 is integrable on $[a, b]$.

2. (10 points) Suppose that $f(x)$ is bounded on $[0,1]$ and integrable on $[\delta,1]$ for all $0<\delta<1$. Prove that $f(x)$ is integrable on $[0,1]$.

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f bounded $\implies \exists M>0$ such that $|f|\leq M$ on $[0,1]$.

Let $\epsilon>0$. Let $\delta=\frac{\epsilon}{4M}$.

Because f integrable on $[\delta,1]$, $\exists$ partition P' of $[\delta,1]$ such that $U(f,P')-L(f,P')<\frac{\epsilon}{2}$.

Let $P=\{0\}\cup P$. Then P is a partition of $[0,1]$.

And

$$U(f,P)-L(f,P)<\delta\cdot 2M+\frac{\epsilon}{2}=\epsilon$$

3. (10 points) Find a number $c > 0$ such that the area under the curve

$$f(x) = \frac{1}{48 + x^3}$$

and above the x-axis from $x = c$ to $x = 2c$ is maximized.

The area in question is

$$F(c) = \int_c^{2c} \frac{1}{48 + x^3}\, dx.$$

f is continuous, so by FTC2 $F(c)$ is differentiable, and

$$\frac{d}{dc} \int_c^{2c} \frac{1}{48 + x^3} = \frac{2}{48 + (2c)^3} - \frac{1}{48 + c^3}$$

This is never undefined for $c > 0$ and is zero when

$$\frac{2}{48 + (2c)^3} = \frac{1}{48 + c^3}$$

or

$$2(48 + c^3) = 48 + 8c^3$$

which occurs when

$$48 = 6c^3$$

or

$$c = 2.$$

For $c < 2$, we have $48 > 6c^3$, and therefore $2(48+c^3) > 48+8c^3$, etc, so $F'(c) > 0$. Similarly, $c > 0 \implies F'(c) < 0$, so $c = 2$ is a maximum by the first derivative test.

4. (10 points) Determine whether the following improper integrals converge or diverge. **You must justify your answer.**

(a) $I = \displaystyle\int_8^\infty \sqrt{\frac{1}{x^{5/3}-1}}\; dx$

First note

$$\sqrt{\frac{1}{x^{5/3}-1}} \geq \sqrt{\frac{1}{x^{5/3}}} = \frac{1}{x^{5/6}}$$

But $5/6 < 1$ means $\int_8^\infty \frac{dx}{x^{5/6}}$ diverges, so, by the Comparison Test, I diverges as well.

(b) $I = \displaystyle\int_1^\infty \frac{\arctan(x)}{x^2+x}\; dx$

First note $\arctan(x) \leq \frac{\pi}{2}$ for all $x \in \mathbb{R}$, so

$$\frac{\arctan(x)}{x^2+x} \leq \frac{\pi}{2} \cdot \frac{1}{x^2+x} \leq \frac{\pi}{2} \cdot \frac{1}{x^2}$$

But

$$\int_1^\infty \frac{\pi}{2} \cdot \frac{1}{x^2} = \frac{\pi}{2} \int_1^\infty \frac{1}{x^2}$$

But $2 > 1$ means $\int_1^\infty \frac{dx}{x^2}$ converges, so $\int_1^\infty \frac{\pi}{2} \cdot \frac{1}{x^2}$ converges, so by the Comparison Test, I converges as well.

5. (10 points) Show that

$$\int_0^{\pi/2} \frac{\sin x + \cos x}{1+x^2} dx \leq \frac{\pi}{\sqrt{2}}.$$

[Hint: What is the maximal value of the numerator?]

Let $f(x) = \sin x + \cos x$. Then $f'(x) = \cos(x) - \sin(x)$, which is never undefined and only equals zero when $\cos(x) = \sin(x) = \sqrt{1-\cos^2(x)}$ or $\cos(x) = \frac{1}{\sqrt{2}}$ (which must therefore also be the value of $\sin(x)$, since they are equal, so $f(x) = \frac{2}{\sqrt{2}}$ at this point – which we "know" is $\frac{\pi}{4}$, but we don't *know* is $\frac{\pi}{4}$). The value of f at the endpoints of $[0, \frac{\pi}{2}]$ is 1, so $f(x) \leq \frac{2}{\sqrt{2}}$ on $[0, \frac{\pi}{2}]$. Thus, since $\arctan(x) \leq \frac{\pi}{2}$ for all $x \in \mathbb{R}$

$$\int_0^{\pi/2} \frac{\sin x + \cos x}{1+x^2} dx \leq \frac{2}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{1+x^2} dx \leq \frac{2}{\sqrt{2}} [\arctan(\pi/2) - \arctan(0)] \leq \frac{2}{\sqrt{2}} \cdot \frac{\pi}{2}$$

which gives the desired bound. You could also use

$$\int_0^{\pi/2} \frac{\sin x + \cos x}{1+x^2} dx \leq \frac{2}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{1+x^2} dx \leq \frac{2}{\sqrt{2}} \int_0^{\pi/2} 1 \, dx \leq \frac{2}{\sqrt{2}} \cdot \frac{\pi}{2}$$

6. (10 points) Calculate:

(a) $\dfrac{d}{dx}\displaystyle\int_{\log(2)}^{2^x} (1+t^2)\, dt$

By FTC1, since $(1+t^2)$ is continuous, we get

$$(1+(2^x)^2)2^x\log(2)$$

(b) For $x>1$, $\dfrac{d}{dx}\big(\log(x)^{\sin(x)}\big)$

By definition,

$$\log(x)^{\sin(x)} = e^{\sin(x)\log\log(x)}$$

so

$$\frac{d}{dx}\big(\log(x)^{\sin(x)}\big) = \frac{d}{dx}e^{\sin(x)\log\log(x)} = e^{\sin(x)\log\log(x)}\left[\cos(x)\log\log x + \sin(x)\frac{1}{x\log x}\right] =$$

$$\log(x)^{\sin(x)}\cdot\left[\cos(x)\log\log x + \sin(x)\frac{1}{x\log x}\right]$$

(c) $\displaystyle\lim_{x\to\infty}\int_1^x \frac{1}{t^3}$

$$\lim_{x\to\infty}\int_1^x \frac{1}{t^3} = \lim_{x\to\infty}\frac{-1}{2}t^{-2}\Big|_1^\infty = \lim_{x\to\infty}\left(\frac{-x^{-2}}{2}+\frac{1}{2}\right) = \frac{1}{2}$$