Part A

1. (20 points) Determine whether each given set S is a subspace of the given vector space V. If so, give a proof; if not, state a property it fails to satisfy.

(a)(5 points) $V = \mathbb{R}^2$, and $S = \left\{(x, y) \in V \,\middle|\, x^2 = y^2\right\}$. $= \{y = x\} \cup \{y = -x\}$

Not closed under addition

$(1,-1), (1,1) \in S$

$(1,-1) + (1,1) = (2,0) \notin S$

Circle final answer. S is a subspace: YES or NO?

NO?

(b)(5 points) $V = \mathbb{R}^2$, and $S = \left\{(x, y) \in V \,\middle|\, x \leq y\right\}$.

Not closed under scaling by (negative) reals

$(1,2) \in S$

but

$(-1)(1,2) = (-1,-2) \notin S$

Circle final answer. S is a subspace: YES or NO?

NO?

(c)(5 points) $V = P_3(\mathbb{R})$, and $S = \left\{ f \in V \,\middle|\, f'(5) = 0 \right\}$.

$f_1'(5)=0, \quad f_2'(5)=0$ then

$(f_1+f_2)'(5) = f_1'(5)+f_2'(5) = 0+0=0$ ✓

$(\alpha f_1)'(5) = \alpha f_1'(5) = \alpha \cdot 0 = 0$ for any scalar α ✓

$0'(5)=0$ ✓

Circle final answer. S is a subspace: YES or NO?

(d)(5 points) $V = P_3(\mathbb{R})$, and $S = \left\{ f \in V \,\middle|\, f'(0) = 5 \right\}$.

$\boxed{0 \notin S}$

OR not closed under addition

$f_1'(0)=5, \; f_2'(0)=5 \rightarrow (f_1+f_2)'(0) = 5+5 = 10 \neq 5$

OR not closed under scaling.

$f_1'(0)=5, \quad (2f_1)'(0) = 10 \neq 5$

Circle final answer. S is a subspace: YES or NO?

2. (16 points) (a)(8 points) Determine if the functions $\{\sin(x), \cos(x), e^x\}$ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ are linearly independent or not. If so, give a justification; if not, explain why not.

cofactor expand along 3rd column

$$\text{Wronskian} = W(x) = \begin{vmatrix} \sin(x) & \cos(x) & e^x \\ \cos(x) & -\sin(x) & e^x \\ -\sin(x) & -\cos(x) & e^x \end{vmatrix}$$

$$= e^x \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} - e^x \begin{vmatrix} \sin(x) & \cos(x) \\ -\sin(x) & -\cos(x) \end{vmatrix} + e^x \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix}$$

$$= e^x(-\cos^2 x - \sin^2 x) - e^x(0) + e^x(-\sin^2 x - \cos^2 x)$$

$$= -2e^x \neq 0 \text{ for any } x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$ [enough to use one specific value]

$\therefore \{\sin(x), \cos(x), e^x\}$ LI in $C^\infty((-\frac{\pi}{2}, \frac{\pi}{2}))$

Circle answer. Are the functions linearly independent: YES or NO?

(b)(8 points) Let $\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} 12 \\ 19 \\ 26 \end{pmatrix}$. Determine if $\vec{b} \in Span\{\vec{a}_1, \vec{a}_2\}$ or not. If so, write $\vec{b}$ explicitly as a linear combination of $\{\vec{a}_1, \vec{a}_2\}$; if not, explain why not.

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 1 & 2 & 19 \\ 1 & 3 & 26 \end{array}\right] \xrightarrow[A_{13}(-1)]{A_{12}(-1)} \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 1 & 7 \\ 0 & 2 & 14 \end{array}\right] \xrightarrow[A_{22}(-2)]{A_{21}(-1)} \begin{array}{ccc} c_1 & c_2 & \\ \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

Consistent with solution $c_1 = 5$, $c_2 = 7$

So $\boxed{\vec{b} = 5\vec{a}_1 + 7\vec{a}_2}$, $\vec{b} \in Span\{\vec{a}_1, \vec{a}_2\}$

Circle answer. $\vec{b} \in Span\{\vec{a}_1, \vec{a}_2\}$: **YES** or **NO**?
If answered YES, write $\vec{b}$ as linear combination here: $5\vec{a}_1 + 7\vec{a}_2$

3. (16 points) Let

$$\vec{v}_1 = 2 + 3x + x^2,\ \vec{v}_2 = 1 + x,\ \text{and } \vec{v}_3 = 1 + 3x + x^2.$$

be three vectors in the vector space $P_2(\mathbb{R})$ of degree ≤ 2 real polynomials.

(a)(8 points) Determine if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $P_2(\mathbb{R})$.

column vectors the 3 polynomials match with under isomorphism $P_2 \cong \mathbb{R}^3$

$$\begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 3 \\ 1 & 0 & 1 \end{vmatrix} \underset{A_{12}(-2)}{\overset{A_{12}(-3)}{=}} \begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \underset{\substack{\text{expand} \\ \text{1st} \\ \text{column}}}{\overset{\text{cofactor}}{=}} (1) \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = \boxed{1} \neq 0$$

As det $\neq 0$ the 3 columns are LI and Span $\mathbb{R}^3$. Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are LI and span $P_2(\mathbb{R})$.

Circle answer. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ **spans** $P_2(\mathbb{R})$: **YES or NO?**

(b)(8 points) Are these three vectors linearly independent? If so, justify why; if not, find an explicit linear dependence between them.

See part (a)

Circle answer. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ **linearly independent? YES or NO?**
If dependent, list linear dependency here: NONE

4. (16 points) Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 3 & 3 & 4 & 4 \\ 0 & 3 & 0 & 4 \end{bmatrix}.$$

(a)(10 points) Find the determinant of A.

$$\begin{vmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 3 & 3 & 4 & 4 \\ 0 & 3 & 0 & 4 \end{vmatrix} \overset{A_{13}(-3)}{=\!=} \begin{vmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 3 & 0 & 4 \end{vmatrix} \underset{\text{1st col}}{\overset{\text{cofactor}}{=\!=}} \begin{vmatrix} 1 & 0 & 2 \\ 0 & -2 & -2 \\ 3 & 0 & 4 \end{vmatrix}$$

$$\| A_{13}(-3)$$

$$\boxed{4} = \begin{vmatrix} -2 & -2 \\ 0 & -2 \end{vmatrix} \underset{\text{1st col}}{\overset{\text{cofactor}}{=\!=}} \begin{vmatrix} 1 & 0 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & -2 \end{vmatrix}$$

$Det(A) =$ 4

(b)(6 points) Is A invertible? Explain why or why not.

Invertible? YES or NO
Explanation: det $\neq 0$

5. (16 points)

Let

$$A = \begin{bmatrix} 4 & 5 & 34 & 17 \\ 1 & 1 & 7 & 3 \\ 3 & 5 & 33 & 19 \end{bmatrix}$$

This matrix has RREF given by:

$$U = \begin{array}{c} \overbrace{x_1 \; x_2}^{\text{bound}} \; \overbrace{x_3 \; x_4}^{\text{free}} \\ \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$x_1 = -x_3 + 2x_4$$
$$x_2 = -6x_3 - 5x_4$$

(a)(6 points) Find a basis for the nullspace of A.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 + 2x_4 \\ -6x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} \quad x_3, x_4 \text{ free}$$

$$= x_3 \begin{bmatrix} -1 \\ -6 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

basis

Basis for nullspace:	$\begin{bmatrix} -1 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$

(b)(6 points) Find a basis for the column space of A.

pivot columns = col 1,2 . Use From A.

Basis for column space:	$\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}$

(c)(4 points) Find the rank and nullity of A.

Rank of A is:	# pivots = 2
Nullity of A is:	# free variables = dim (Nullspace) = 2

6. (16 points) Let $S_1 = \{A \in M_3(\mathbb{R}) \mid A + A^T = 0\}$ and $S_2 = \{A \in M_4(\mathbb{R}) \mid A = 2A^T\}$. Here $M_n(\mathbb{R})$ denotes the vector space of $n \times n$ real matrices.

(a)(8 points) Find a basis for S_1.

$$S_1 = \left\{ A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

Basis for S_1: $\left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right)$

(b)(4 points) Determine the dimension of S_1.

Dimension of S_1: 3

(c)(4 points) Determine the dimension of S_2.

$$A = 2A^T \implies A = 2(2A^T)^T = 4A$$
$$\implies A = 4A$$
$$\implies 3A = 0$$
$$\implies A = 0$$

$S_2 = \{0\}$ Basis $= \{\ \}$, dim $= 0$

Dimension of S_2: 0