Part A

1. (16 points) Find the solution to the initial value problem

$$xy' + y = y^2, \quad y(3) = -1$$

$$x\frac{dy}{dx} = y^2 - y = y(y-1)$$ (Equilibrium solutions are $y \equiv 0$, $y \equiv 1$)

(For $y \neq 0, 1$, $x \neq 0$) $$\int \frac{dy}{y(y-1)} = \int \frac{dx}{x} + C$$ ("Separation of variables")

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} \longrightarrow 1 = A(y-1) + B(y) \xrightarrow{y=0} A = -1, \quad \xrightarrow{y=1} B = 1$$ ("Partial Fractions")

$$\therefore \int \frac{1}{y-1}\,dy - \int \frac{1}{y}\,dy = \int \frac{dx}{x} + C$$

$$\therefore \ln|y-1| - \ln|y| = \ln|x| + C$$

$$\ln\left|\frac{y-1}{y}\right| = \ln|x| + C$$ exponentiate

$$\left|\frac{y-1}{y}\right| = e^C|x|$$

$$\frac{y-1}{y} = Ax \quad (A = \pm e^C)$$

$$y - 1 = Axy$$

$$y(1-Ax) = y - Axy = 1$$

$$\therefore y = \frac{1}{1-Ax}$$

$y(3) = -1$ so $$-1 = \frac{1}{1-A(3)} \longrightarrow 3A - 1 = 1 \longrightarrow A = \frac{2}{3}$$

Answer: $y(x) = \dfrac{1}{1 - \frac{2}{3}x}$

2. (16 points) Find the general solution of

$$t^3 y' = 3t^2 y + 15, \quad \text{for } t > 0$$

$$y' = \frac{3}{t} y + \frac{15}{t^3} \quad \text{so} \quad y' \underbrace{-\frac{3}{t}}_{=P} y = \underbrace{\frac{15}{t^3}}_{=Q}$$

Integration factor: $e^{\int P(t)dt} = e^{-\int \frac{3}{t}dt} = e^{-3\ln(t)} = e^{\ln(t^{-3})} = \frac{1}{t^3}$

Multiply both sides by integration factor yields:

$$\left(y\left(\frac{1}{t^3}\right)\right)' = \frac{15}{t^3}\frac{1}{t^3} = \frac{15}{t^6}$$

Integrate

$$y\left(\frac{1}{t^3}\right) = \int \frac{15}{t^6}dt + C$$

$$\left(\frac{1}{t^3}\right)y = \frac{15t^{-5}}{-5} + C$$

$$y = \frac{-3}{t^2} + Ct^3$$

Answer: $y(t) = -\frac{3}{t^2} + Ct^3$

3. (12 points) Find all values of r such that $y = t^r e^{2t}$ is a solution to the following differential equation:

$$y'' - 4y' + 4y = 0$$

Justify your answer.

$$y = t^r e^{2t}$$

$$y' = (t^r e^{2t})' = (t^r)' e^{2t} + t^r (e^{2t})'$$

$$\boxed{y' = r t^{r-1} e^{2t} + 2t^r e^{2t}}$$

$$y'' = r(t^{r-1} e^{2t})' + 2(t^r e^{2t})'$$

$$y'' = r(r-1)t^{r-2} e^{2t} + r t^{r-1} 2e^{2t}$$
$$+ 2rt^{r-1} e^{2t} + 2t^r 2e^{2t}$$

$$\boxed{y'' = r(r-1)t^{r-2} e^{2t} + 4r t^{r-1} e^{2t} + 4t^r e^{2t}}$$

Plug into ODE.

$$\left(r(r-1)t^{r-2}e^{2t} + \cancel{4rt^{r-1}e^{2t}} + \cancel{4t^r e^{2t}}\right) - 4\left(\cancel{rt^{r-1}e^{2t}} + \cancel{2t^r e^{2t}}\right) + 4\left(\cancel{t^r e^{2t}}\right) = 0$$

$$r(r-1)t^{r-2}e^{2t} = 0$$

As must hold for all t, only works if $r(r-1) = 0$

i.e. $r = 0$ or $r = 1$

Answer: Values of r that work are: 0 or 1

4. (16 points) A tank contains 300 L of solution with a concentration of 0.3 kg of salt per L initially (at $t = 0$). A solution of variable concentration $c_{in}(t)$ enters the tank through an inflow pipe at a rate of 24 L/min. The concentration $c_{in}(t)$ is not constant but instead proportional to time t (in minutes) with $c_{in}(5) = 0.5$ kg of salt per L. The solution is mixed and drains from the tank at the rate of 20 L/min.

(a) Find a formula for the volume $V(t)$ of solution in the tank at time t. (Assume the tank never fills completely during the timespan of the problem.)

$$\frac{dV}{dt} = V_{in} - V_{out} = (24-20)\ L/min \rightarrow \frac{dV}{dt} = 4$$
$$\rightarrow V = 4t + C$$
$$\rightarrow V = 4t + 300 \ (\text{as } t=0 \text{ has } V=0)$$

$V(t) = (4t + 300)$ (Units L)

(b) Find $c_{in}(t)$. $C_{in} = kt$ (As C_{in} is "proportional" to t) k = some constant

$C_{in}(5) = 0.5 \rightarrow 0.5 = k5 \rightarrow k = 0.1$

$c_{in}(t) =$ $(0.1)t$ (Units kg/L)

(c) Write down a differential equation for the amount $A(t)$ of salt in the tank, and state initial conditions for your differential equation. **DO NOT SOLVE THE EQUATION.**

$$\frac{dA}{dt} = (C_{in})(V_{rate\,in}) - (C_{out})(V_{rate\,out}) = C_{in}(24) - \frac{A}{V}(20)$$
$$= (24)(0.1t) - \frac{A}{4t+300}(20) = 2.4t - \frac{20A}{4t+300}$$

At time $t=0$, $C_{tank} = 0.3 kg/L = \frac{A(0)}{V(0)} = \frac{A(0)}{300L} \rightarrow A(0) = 90\ kg$

Final form of Differential equation: $\dfrac{dA}{dt} + \dfrac{20A}{4t+300} = 2.4t$, $A(0) = 90kg$

5. (20 points)

(a) Use elementary row operations to put the matrix

$$\mathbb{A} = \begin{bmatrix} 3 & 4 & -8 \\ 5 & 2 & -18 \\ 1 & 6 & 2 \end{bmatrix}$$

into **reduced** row echelon form. Indicate which row operations are being used at each step.

$$\begin{bmatrix} 3 & 4 & -8 \\ 5 & 2 & -18 \\ 1 & 6 & 2 \end{bmatrix} \xrightarrow[A_{32}(-5)]{A_{31}(-3)} \begin{bmatrix} 0 & -14 & -14 \\ 0 & -28 & -28 \\ 1 & 6 & 2 \end{bmatrix} \xrightarrow{M_1(-\frac{1}{14})} \begin{bmatrix} 0 & 1 & 1 \\ 0 & -28 & -28 \\ 1 & 6 & 2 \end{bmatrix}$$

$$\xrightarrow[A_{13}(-6)]{A_{12}(28)} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -4 \end{bmatrix} \xrightarrow[\text{followed by } P_{23}]{P_{13}} \underbrace{\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{RREF}}$$

(b) Find the rank of the matrix A and explain how you know this is the rank of $\mathbb{A}$.

Rank: 2

Explanation: # of pivot/leading 1's in RREF of $\mathbb{A}$.

(c) Describe all solutions to the homogeneous system $\mathbb{A}\vec{x} = \vec{0}$.

$$\mathbb{A}\vec{x} = \vec{0}\ ; \quad \left[\begin{array}{ccc|c} 3 & 4 & -8 & 0 \\ 5 & 2 & -18 & 0 \\ 1 & 6 & 2 & 0 \end{array}\right] \xrightarrow[\text{in (a)}]{\text{row ops}} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\{x_1, x_2\}$ = Bound variables, $\{x_3\}$ = free variable

$$x_1 = 4x_3$$
$$x_2 = -x_3$$

$$\therefore \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4t \\ -t \\ t \end{pmatrix},\ t \text{ free}$$

$$\vec{x} = t\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix},\ t \text{ free}$$

Solution(s): $t\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$ free.

(d) Now suppose that $\mathbb{B}$ is a 4×5 matrix with rank$(\mathbb{B}) = 4$. How many solutions are there to the system $\mathbb{B}\vec{x} = \vec{0}$? Justify your answer.

Number of solutions: ∞ Solutions

Justification: $B\vec{x} = \vec{0}$ always has at least one solution $\vec{x} = \vec{0}$. As rank$(\mathbb{B}) = 4$ $<$ # columns of $\mathbb{B} = 5$ some column missing leading 1 $\rightarrow$ free var. $\rightarrow$ ∞ sols.

6. (20 points) For parts (a) and (b), use Gaussian (or Gauss-Jordan) elimination to find all solutions to the given system or show that no solutions exist. Show all steps.

(a)

$$x_1 + 2x_2 + 3x_3 = 1$$
$$3x_1 + 5x_2 + 5x_3 = 4$$
$$2x_1 + 3x_2 + x_3 = -3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 5 & 5 & 4 \\ 2 & 3 & 1 & -3 \end{array}\right] \xrightarrow[A_{13}(-2)]{A_{12}(-3)} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 1 \\ 0 & -1 & -5 & -5 \end{array}\right]$$

$$\downarrow M_2(-1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 3 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & -1 & -6 \end{array}\right] \xleftarrow[A_{21}(-2)]{A_{23}(1)} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & -1 & -5 & -5 \end{array}\right]$$

$$\downarrow M_3(-1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 3 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 6 \end{array}\right] \xrightarrow[A_{31}(5)]{A_{32}(-4)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 33 \\ 0 & 1 & 0 & -25 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

Solution(s): $(X_1 = 33,\ X_2 = -25,\ X_3 = 6)$: unique solution for $\vec{x}$.

(b) $\mathbb{A}\vec{x} = \vec{b}$ where

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 0 & 5 & 1 & 2 \\ 0 & 2 & 1 & 4 \end{array}\right] \xrightarrow[A_{13}(-2)]{A_{12}(-5)} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 0 & 0 & 6 & -8 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{M_3(\frac{1}{3})} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 0 & 0 & 6 & -8 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[A_{31}(1)]{A_{32}(-6)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\text{followed by rearrange rows}]{M_2(-\frac{1}{8})} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

"$0=1$" No solution

Solution(s): No solution.