# MATH 165 Linear Algebra and Differential Equations

Final Exam

December 18, 2022

**Name:** Solutions

**UR ID:** ____________________

**Circle your instructor's name:**

Salur    Hopper    Chaves    Han

**Instructions for the midterm you will be taking:**

- The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden. If you have questions, get the attention of your exam proctor; otherwise no communication is allowed during the exam.
- Show your work! You may not receive full credit for a correct answer if insufficient work or insufficient justification is given. In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$. Be sure to include units when applicable!
- The exam is 2 hour, 30 minutes and is worth 200 points.

COPY THE HONOR PLEDGE AND SIGN (Cursive is not required)

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE: ____________________

| Part A | | |
|---|---|---|
| QUESTION | VALUE | SCORE |
| 1 | 20 | |
| 2 | 20 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 15 | |
| TOTAL | 100 | |

| Part B | | |
|---|---|---|
| QUESTION | VALUE | SCORE |
| 7 | 15 | |
| 8 | 15 | |
| 9 | 20 | |
| 10 | 15 | |
| 11 | 15 | |
| 12 | 20 | |
| TOTAL | 100 | |

**Part A**

**1. (20 pts)**

(a) Find the general solution of the differential equation

$$t^2\frac{dy}{dt} + ty = t^2 e^{t^2}. \quad \text{(may assume } t>0\text{)}$$

(Your solution should be explicit, i.e. solved for $y$.)

Normal form. $\frac{dy}{dt} + \frac{1}{t}y = e^{t^2}$

Integrating factor: $e^{\int \frac{1}{t}dt} = e^{\ln t} = t$

$\Rightarrow \underbrace{t\frac{dy}{dt} + y}_{\frac{d}{dt}(ty)} = te^{t^2}$ $\Rightarrow ty = \int te^{t^2} = \frac{1}{2}e^{t^2} + c$

$\Rightarrow y(t) = \frac{1}{t}\left(\frac{1}{2}e^{t^2} + c\right)$

**Solution:** $y(t) = \frac{1}{t}\left(\frac{1}{2}e^{t^2} + c\right)$

(b) Solve the initial value problem

$$\frac{dy}{dx} = \frac{x\sin(x)}{6y^2}, \qquad y(0) = -2.$$

$$\int 6y^2\,dy = \int x\sin x\,dx \qquad u = x \quad v = -\cos x \qquad du = dx \quad dv = \sin x\,dx$$

$$2y^3 = -x\cos x - \int -\cos x\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

$$x=0 \quad y=2 \Rightarrow \quad 2(-2)^3 = C \quad \Rightarrow C = -16.$$

$$2y^3 = -x\cos x + \sin x - 16$$

$$\Rightarrow y = \sqrt[3]{\frac{1}{2}\left(-x\cos x + \sin x - 16\right)}$$

**Solution:** $y = \sqrt[3]{\frac{1}{2}\left(-x\cos x + \sin x - 16\right)}$

Population Growth will not be covered this semester!

**2. (20 pts)** Suppose that the population of a city increases with rate proportional to the population itself. The population was 20 thousand in the year 2000 and 24 thousand in the year 2020.

(a) Find the population $P(t)$ (in thousands) in the year $t$.

| |
|---|
| $P(t) =$ |

(b) The population is expected to be $Y$ thousand in the year 2040. Find the value of $Y$. (Your answer should be an explicit number without constants such as $e$.)

| |
|---|
| $Y =$ |

**3. (15 pts)** Use either Gaussian or Gauss-Jordan elimination to find the solution(s) to the following system of linear equations:

$$x + 2y + z = 3$$
$$2x - y + 2z = 1$$
$$3x + y + 3z = 4$$

Write your answer in vector format. If the system is inconsistent, write "inconsistent". Justify your answer. (No credit will be given for a method not based on either Gaussian or Gauss-Jordan elimination)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 2 & 1 \\ 3 & 1 & 3 & 4 \end{array}\right] \underset{A_{13}(-3)}{\overset{A_{12}(-2)}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & 0 & -5 \\ 0 & -5 & 0 & -5 \end{array}\right] \overset{M_2(-\frac{1}{5})}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -5 & 0 & -5 \end{array}\right]$$

$$\overset{A_{23}(5)}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3$: free

$x_1 + 2x_2 + x_3 = 3 \Rightarrow x_1 = 3 - 2x_2 - x_3$

$x_2 = 1$

$= 1 - x_3$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 1 \\ x_3 \end{bmatrix}$$

**Answer:** $\vec{x} = \begin{bmatrix} 1 - x_3 \\ 1 \\ x_3 \end{bmatrix} \left( = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$

**4. (15 pts)** Consider the following matrix.

$$A = \begin{pmatrix} -1 & -5 & -2 & 2 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Determine a basis and the dimension of rowspace($A$). Justify your answer.

A is already in a row echelon form.

⇒ Nonzero rows of A form a basis

basis

**Answer:** $\{[-1\ -5\ -2\ 2\ 3], [0\ 0\ 1\ -1\ -3], [0\ 0\ 0\ 0\ 1]\}$

dimension = 3

(b) Determine a basis and the dimension of colspace($A$). Justify your answer.

Columns w/ leading 1 form a basis.

basis

**Answer:** $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$ dimension = 3

(c) Determine the dimension of nullspace($A$). Justify your answer.

Rank-Nullity theorem: rank(A) + nullity(A) = number of columns = 5

rank(A) ↓ 3

⇒ dim(nullspace(A)) = nullity(A) = 5 − 3 = 2

**Answer:** 2

**5. (15 pts)** Find the values of $k$ so that the following matrix is NOT invertible.

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 0 & 3k \\ -1 & 3 & k^2 - 4 \end{pmatrix}$$

$\Updownarrow$
det $= 0$

$\det(A) = (-9k) + (-9) - 9(k^2 - 4) - 9k$

$= -9k^2 - 18k + 27$

$= -9(k^2 + 2k - 3) = -9(k+3)(k-1)$

**Answer:** $k = -3$, $k = 1$.

**6. (15 pts)** For each set of vectors $S$ in the vector space $V$ , decide which category $S$ fits into (*You must show all your work to receive credit*):

I. $S$ is linearly independent and spans $V$.

II. $S$ is linearly independent and does not span $V$.

III. $S$ is linearly dependent and spans $V$.

IV. $S$ is linearly dependent and does not span $V$

(a) $V = \mathbb{R}^3$, $S = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}$. $\to$ 3 vectors in $\mathbb{R}^3$

$$\det \begin{bmatrix} 0 & -2 & 1 \\ 2 & 6 & 0 \\ 1 & -5 & 4 \end{bmatrix} = -10 - (-16) - 6 = 0 \Rightarrow \text{rank} < 3$$

$\Rightarrow$ LD & Not span.

**Answer:** IV

(b) $V = P_2(\mathbb{R})$, $S = \{x^2+1,\ x+2,\ x^2-2x-3\}$

↕ identify

$\mathbb{R}^3$ $\{(1,0,1), (0,1,2), (1,-2,-3)\}$ 3 vectors in $\mathbb{R}^3$

$\det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 2 & -3 \end{bmatrix} = -3 - 1 - (-4) = 0 \Rightarrow \text{rank} < 3$

$\Rightarrow$ LD & Not span

**Answer:** IV.

(c) $V = M_{2\times 2}(\mathbb{R})$, $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$.

↕ identify

$\mathbb{R}^4$ $\{(1,0,0,1), (1,0,-1,0), (1,1,1,1)\}$ 3 vectors in $\mathbb{R}^4$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ rank $=3 \Rightarrow$ LI, not span.

**Answer:** II.

**Part B**

**7. (15 pts)** Let $v = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Consider the subspace $W$ of $V = M_{2\times 2}(\mathbb{R})$ consisting of the matrices

$$W = \{A \in V \mid Av = 0\}.$$

Find a basis for $W$ and determine $\dim(W)$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow Av = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} a-3b \\ c-3d \end{bmatrix}$

so $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V : a-3b = c-3d = 0 \right\}$

$\Downarrow$ $a=3b,\ c=3d$

$= \left\{ \begin{bmatrix} 3b & b \\ 3d & d \end{bmatrix} \in V : b, d \in \mathbb{R} \right\}$

$\Downarrow$

$b\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}.$

$\Rightarrow \left\{ \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} \right\}$ spans $W$, also LI $\left( a\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = b = 0 \right)$

$\rightarrow$ basis for $W$.

$\dim = 2$

**Answer:** basis $\left\{ \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} \right\}$. $\dim(W) = 2$

**8. (15 pts)** Let $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(f) = \begin{pmatrix} f(1) & f'(1) \\ 2f(1) & 2f(1) - f'(1) \end{pmatrix}.$$

(a) Find a basis for the kernel of $T$.

$$\text{Ker}(T) = \left\{ f \in P_2(\mathbb{R}) : \begin{bmatrix} f(1) & f'(1) \\ 2f(1) & 2f(1)-f'(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \{ f \in P_2(\mathbb{R}) : f(1) = f'(1) = 0 \}$$

Let $f(x) = ax^2 + bx + c$ $\Rightarrow$ $f(1) = a+b+c$

$f'(x) = 2ax + b$ $\quad$ $f'(1) = 2a+b$.

$$\rightarrow \text{Ker}(T) = \{ ax^2+bx+c \in P_2(\mathbb{R}) : a+b+c = 2a+b = 0 \}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 2 & 1 & 0 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & -1 & -2 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

$a = c$, $b = -2c$

$$= \{ cx^2 - 2cx + c : c \in \mathbb{R} \}$$

$cx^2 - 2cx + c = c(x^2 - 2x + 1)$

$$= \text{Span}(\{ x^2 - 2x + 1 \})$$

**Answer:** $\text{Span}(\{x^2 - 2x + 1\})$

(b) Determine the dimensions of the kernel and range (i.e. image) of $T$. Justify your answer.

$\dim(\ker(T)) = 1$ from Part (a)

$\dim(\text{Rng}(T)) = \dim P_2(\mathbb{R}) - \dim(\ker(T))$ (General R-N Theorem)

$= 3 - 1 = 2$

**Answer:** $\dim(\ker(T)) = 1$
$\dim(\text{Rng}(T)) = 2$

(c) Find a basis for the range of $T$.

$f(x) = ax^2 + bx + c$

$$\Rightarrow T(f) = \begin{bmatrix} f(1) & f'(1) \\ 2f(1) & 2f(1) - f'(1) \end{bmatrix} = \begin{bmatrix} a+b+c & 2a+b \\ 2(a+b+c) & 2(a+b+c) - (2a+b) \end{bmatrix}$$

$$= \begin{bmatrix} a+b+c & 2a+b \\ 2a+2b+2c & b+2c \end{bmatrix} = a\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

$\left\{\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\right\}$ spanning set for $\text{Rng}(T)$, but not a basis ($\dim \text{Rng}(T) = 2$)

In fact, $-\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow \left\{\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\right\}$ is also a spanning set, and a basis for $\text{Rng}(T)$

**Answer:** $\left\{\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\right\}$.

Alternatively:

$T(f) = f(1)\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} + f'(1)\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and $f(1), f'(1)$ can be any real numbers $\to$ $\left\{\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right\}$ is a spanning set so a basis.

**9. (20 pts)** Consider the following matrix.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(a) Determine all eigenvalues of $A$ with their multiplicities.

$$\det(A - \lambda I_3) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ -2 & -1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda) \det \begin{bmatrix} 2-\lambda & 1 \\ -2 & -1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left((2-\lambda)(-1-\lambda) - (-2)\right)$$

$$= (1-\lambda)\left(\lambda^2 - \lambda - 2 - (-2)\right)$$

$$= (1-\lambda)(\lambda^2 - \lambda)$$

$$= -\lambda(\lambda-1)^2.$$

**Answer:** $\lambda = 0$ multiplicity 1, $\lambda = 1$ multiplicity 2

(b) For each eigenvalue $\lambda$ of $A$, find a basis for the eigenspace $E_\lambda$.

$\lambda = 0$: $A - \lambda I_3 = \begin{bmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_3$
$x_2 = -2x_3$
$x_3$: free

$\rightarrow E_0 = \left\{ \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

$\lambda = 1$: $A - \lambda I_3 = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -x_2$
$x_2, x_3$: free

$E_1 = \left\{ \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

**Answer:** Basis for $E_0$: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$
" $E_1$: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) Determine whether $A$ is defective or nondefective. You should justify your answer.

dim of Eigenspace = multiplicity for every eigenvalue.

**Answer:** Nondefective

**10. (15 pts)** Find the general solution to the differential equation

$$y'' - 5y' + 6y = 1 + 6t + 5e^{-2t}$$

Aux. poly : $P(r) = r^2 - 5r + 6 = (r-2)(r-3)$

$\Rightarrow$ Complementary function : $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$.

Annihilator of $\underbrace{1+6t}_{D^2} + \underbrace{5e^{-2t}}_{D+2}$ : $D^2(D+2)$.

New homogeneous DE: $D^2(D+2)(D-2)(D-3)\, y = 0$

$\rightarrow$ General solution: $y = \underbrace{c_1 e^{2t} + c_2 e^{3t}}_{y_c} + c_3 e^{-2t} + c_4 t + c_5$

$\rightarrow$ Trial solution: $y = A_0 e^{-2t} + A_1 t + A_2$

$y' = -2A_0 e^{-2t} + A_1$

$y'' = 4A_0 e^{-2t}$

$y'' - 5y' + 6y = 4A_0 e^{-2t} - 5(-2A_0 e^{-2t} + A_1) + 6(A_0 e^{-2t} + A_1 t + A_2)$

$= \underset{= 5}{\underline{20 A_0}} e^{-2t} + \underset{= 6}{\underline{6A_1}} t + (\underset{= 1}{\underline{-5A_1 + 6A_2}})$

$A_0 = \frac{1}{4}$, $A_1 = 1$, $A_2 = 1$ $\Rightarrow$ Particular solution : $y_p = \frac{1}{4} e^{-2t} + t + 1$.

**Solution:** $y_p + y_c = \frac{1}{4} e^{-2t} + t + 1 + c_1 e^{2t} + c_2 e^{3t}$

**11. (15 pts)** Solve the initial value problem

$$y'' + 36y = 0, \qquad y\left(\frac{\pi}{6}\right) = 3, \quad y'\left(\frac{\pi}{6}\right) = -2.$$

Aux. poly. $p(r) = r^2 + 36 \to$ Roots $r = \pm 6i$

General solution: $y = C_1 \cos(6x) + C_2 \sin(6x)$, $y' = -6C_1 \sin(6x) + 6C_2 \cos(6x)$

$y(\frac{\pi}{6}) = 3 \Rightarrow C_1 \cos(\pi) + C_2 \sin(\pi) = -C_1 = 3 \to C_1 = -3$

$y'(\frac{\pi}{6}) = -2 \Rightarrow -6C_1 \sin(\pi) + 6C_2 \cos(\pi) = -6C_2 = -2 \to C_2 = \frac{1}{3}$

Particular solution: $y = -3\cos(6x) + \frac{1}{3}\sin(6x)$

**Solution:** $y = -3\cos(6x) + \frac{1}{3}\sin(6x)$.

**12. (20 pts)** Suppose $x_1(t)$ and $x_2(t)$ are differentiable functions satisfying the system of differential equations

$$x_1' = -2x_1 + 17x_2$$
$$x_2' = -x_1 - 4x_2.$$

(a) Specify a vector $\mathbf{x}$ and matrix $A$ such that vector formulation of the above system is $\mathbf{x}' = A\mathbf{x}$.

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $A = \begin{bmatrix} -2 & 17 \\ -1 & -4 \end{bmatrix}$

(b) Determine the particular solution of $\mathbf{x}' = A\mathbf{x}$ where $\mathbf{x}(0) = \begin{pmatrix} 17 \\ -1 \end{pmatrix}$. (Your answer may be in vector form.)

$$\det(A-\lambda I_2) = \det\begin{bmatrix} -2-\lambda & 17 \\ -1 & -4-\lambda \end{bmatrix} = (-2-\lambda)(-4-\lambda)-(-17)$$
$$=(\lambda^2+6\lambda+8)+17 = \lambda^2+6\lambda+25$$
$$\text{Roots: } -3 \pm \sqrt{9-25} = -3 \pm 4i$$

$$A-(-3+4i)I_2 = \begin{bmatrix} 1-4i & 17 \\ -1 & -1-4i \end{bmatrix} \sim \begin{bmatrix} 1 & 1+4i \\ 0 & 0 \end{bmatrix}$$

$$E_{-3+4i} = \text{Span}\left\{\begin{bmatrix} -1-4i \\ 1 \end{bmatrix}\right\}. \qquad e^{(-3+4i)t} = e^{-3t}(\cos(4t)+\sin(4t)i)$$

$$\begin{bmatrix} -1-4i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} i$$

$$e^{-3t}(\cos(4t)+\sin(4t)i)\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} i\right) = e^{-3t}\left(\left(\cos(4t)\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \sin(4t)\begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) + \left(\cos(4t)\begin{bmatrix} -4 \\ 0 \end{bmatrix} + \sin(4t)\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)i\right)$$

$$= e^{-3t}\begin{bmatrix} -\cos(4t)+4\sin(4t) \\ \cos(4t) \end{bmatrix} + e^{-3t}\begin{bmatrix} -4\cos(4t)-\sin(4t) \\ \sin(4t) \end{bmatrix} i$$

**Solution:** $\vec{x}(t) = -e^{-3t}\begin{bmatrix} -\cos(4t)+4\sin(4t) \\ \cos(4t) \end{bmatrix} - 4e^{-3t}\begin{bmatrix} -4\cos(4t)-\sin(4t) \\ \sin 4t \end{bmatrix} = e^{-3t}\begin{bmatrix} 17\cos(4t) \\ -\cos(4t)-4\sin(4t) \end{bmatrix}$



See next page!

**THIS PAGE INTENTIONALLY LEFT BLANK. Do not tear this page off.** You may use this page if you run out of space. Make sure to label your solution(s) on this page and also include a note on the problem page(s) telling the grader(s) to look for your work here.

12(b) General solution: $\vec{x}(t) = c_1 e^{-3t}\begin{bmatrix} -\cos(4t) + 4\sin(4t) \\ \cos(4t) \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} -4\cos(4t) - \sin(4t) \\ \sin 4t \end{bmatrix}$

$\qquad\qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$\vec{x}(0) = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} -c_1 - 4c_2 = 17 \\ c_1 \qquad = -1 \end{matrix} \Rightarrow c_1 = -1,\ c_2 = -4$