MATH 165 Linear Algebra and Differential Equations

Midterm 2

November 8, 2022

Name: Solutions

UR ID: ____________________

Circle your instructor's name:

Salur Hopper Chaves Han

Instructions for the midterm you will be taking:

- The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden. If you have questions, get the attention of your exam proctor; otherwise no communication is allowed during the exam.
- Show your work! You may not receive full credit for a correct answer if insufficient work or insufficient justification is given. In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$. Be sure to include units when applicable!
- The exam is 1 hour, 15 minutes and is worth 100 points.

COPY THE HONOR PLEDGE AND SIGN (Cursive is not required)

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

YOUR SIGNATURE: ____________________

QUESTION	VALUE	SCORE
1	15	
2	20	
3	15	
4	15	
5	20	
6	15	
TOTAL	100	

1. (15 pts) Consider the following system of linear equations, where k is a real number.

$$\begin{aligned} -x_1 + x_2 - 4x_3 &= -3 \\ 2x_1 - 3x_2 + 5x_3 &= 5 \\ -2x_1 + 4x_2 + kx_3 &= -2 \end{aligned}$$

(a) Find value(s) of k such that the system is inconsistent. Justify your answer.

$$\left[\begin{array}{ccc|c} -1 & 1 & -4 & -3 \\ 2 & -3 & 5 & 5 \\ -2 & 4 & k & -2 \end{array}\right] \underset{A_{13}(-2)}{\overset{A_{12}(2)}{\sim}} \left[\begin{array}{ccc|c} -1 & 1 & -4 & -3 \\ 0 & -1 & -3 & -1 \\ 0 & 2 & k+8 & 4 \end{array}\right] \underset{M_2(-1)}{\overset{M_1(-1)}{\sim}} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & k+8 & 4 \end{array}\right]$$

$$\overset{A_{23}(-2)}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & k+2 & 2 \end{array}\right]$$

If $k=-2$, the system is inconsistent
otherwise it is consistent (unique solution)

Value(s) of k: $k=-2$.

(b) Find all solutions of the system when $k = -1$. If the system is inconsistent, write "inconsistent." If the system is consistent, write your answer in a vector format.

$$k=-1 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 4 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{aligned} x_1 &= -10 \\ x_2 &= -5 \\ x_3 &= 2 \end{aligned}$$

Solution(s): $\vec{x} = \begin{bmatrix} -10 \\ -5 \\ 2 \end{bmatrix}$

2. (20 pts) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 2 \\ 3 & 3 & 2 \end{pmatrix}.$$

(a) Compute $\det(\mathbf{A})$.

$$\begin{array}{ccccc} 2 & 1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 & -1 \\ 3 & 3 & 2 & 3 & 3 \end{array}$$

$$-3 \quad 12 \quad 0 \quad -4 \quad 6 \quad 0$$

$$-4+6+0-(-3)-12-0$$
$$=-7$$

$\det(\mathbf{A}) = -7$

(b) Compute $\mathbf{A}^{-1}$.

(i)
$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \underset{A_{13}(-1)}{\sim} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & -1 & 0 & 1 \end{array}\right] \underset{P_{13}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\underset{A_{13}(-2)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -1 & 3 & 0 & -2 \end{array}\right] \underset{M_2(-1)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & -3 & -1 & 3 & 0 & -2 \end{array}\right] \underset{A_{23}(3)}{\overset{A_{21}(-2)}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & 2 & 1 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & -7 & 3 & -3 & -2 \end{array}\right]$$

$$\underset{M_3(-\frac{1}{7})}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & 2 & 1 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{7} & \frac{3}{7} & \frac{2}{7} \end{array}\right] \underset{A_{32}(2)}{\overset{A_{31}(-5)}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{7} & -\frac{1}{7} & -\frac{3}{7} \\ 0 & 1 & 0 & -\frac{6}{7} & -\frac{1}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -\frac{3}{7} & \frac{3}{7} & \frac{2}{7} \end{array}\right]$$

$$\mathbf{A}^{-1} = -\frac{1}{7}\begin{bmatrix} -8 & 1 & 3 \\ 6 & 1 & -4 \\ 3 & -3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{8}{7} & -\frac{1}{7} & -\frac{3}{7} \\ -\frac{6}{7} & -\frac{1}{7} & \frac{4}{7} \\ -\frac{3}{7} & \frac{3}{7} & \frac{2}{7} \end{bmatrix}$$

(ii) Minors: $\begin{bmatrix} -8 & -6 & 3 \\ -1 & 1 & 3 \\ 3 & 4 & -2 \end{bmatrix}$ Cofactors: $\begin{bmatrix} -8 & 6 & 3 \\ 1 & 1 & -3 \\ 3 & -4 & -2 \end{bmatrix}$ Adjoint: $\begin{bmatrix} -8 & 1 & 3 \\ 6 & 1 & -4 \\ 3 & -3 & -2 \end{bmatrix}$

⊛ This notation implies determinants

3. (15 pts) Suppose that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4.$$

(a) Find $\begin{vmatrix} -a & -b & -c \\ d+g & e+h & f+i \\ g & h & i \end{vmatrix} = - \begin{vmatrix} a & b & c \\ d+g & e+h & f+i \\ g & h & i \end{vmatrix} = - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -4$

(b) Find $\begin{vmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{vmatrix} = 2 \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = -2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -8$

(c) Find $\begin{vmatrix} a+3g & b+3h & c+3i \\ g & h & i \\ d-g & e-h & f-i \end{vmatrix} = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -4$

4. (15 pts) For each of the following vector spaces V and subsets S, determine whether S is a subspace of V. Show all work; unjustified answers will not receive credit.

(a) $V = \mathbb{R}^3$ and $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x + y + z > -1 \right\}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in S \quad (1+1+1 > -1)$ but $-1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \notin S \quad (-1-1-1 < -1)$

Not closed under scalar multiplication

Is S a subspace of V?	**YES**	**NO**

(b) $V = M_{2\times 2}(\mathbb{R})$, the set of 2×2 matrices with entries in $\mathbb{R}$, and

$$S = \{A \in M_{2\times 2}(\mathbb{R}) \mid 3A^T = 0\}$$

$0_{2\times 2} \in S \quad (3 0_{2\times 2}^T = 0_{2\times 2})$

$A, B \in S \Rightarrow 3A^T = 3B^T = 0_{2\times 2} \Rightarrow \quad 3(A+B)^T = 3(A^T + B^T) = 3A^T + 3B^T = 0_{2\times 2} + 0_{2\times 2} = 0_{2\times 2} \Rightarrow A+B \in S$

$A \in S, k \in \mathbb{R} \Rightarrow 3A^T = 0_{2\times 2} \Rightarrow 3(kA)^T = (3k)A^T = k(3A^T) = k0_{2\times 2} = 0_{2\times 2} \Rightarrow kA \in S$

Is S a subspace of V?	**YES**	**NO**

(c) $V = P_3(\mathbb{R})$, the set of polynomials of degree ≤ 3 with coefficients in $\mathbb{R}$, and

$$S = \{p \in P_3(\mathbb{R}) \mid p'(5) = 1\},$$

where $p'(x)$ denotes the first derivative of $p(x)$

Let $p(x)$ be the zero polynomial, i.e. $p(x) = 0$

$\Rightarrow p'(x) = 0$, so $p'(5) = 0 \neq 1 \quad \Rightarrow p(x) \notin S$.

Fails the zero vector check.

Is S a subspace of V?	**YES**	**NO**

5. (20 pts)

(a) Consider two vectors

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 3 \\ -5 \\ k \end{pmatrix}.$$

Find value(s) of k such that

$$\mathbf{v_3} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

is in the linear span of $\{\mathbf{v_1}, \mathbf{v_2}\}$.

$V_3 \in \text{Span}\{V_1, V_2\}$ if and only if $\begin{bmatrix} 1 & 3 \\ -2 & -5 \\ 2 & k \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is consistent.

$$\left[\begin{array}{cc|c} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 2 & k & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & k-6 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & k-4 \end{array}\right]$$

Consistent if and only if $k-4=0 \Leftrightarrow k=4$

Value(s) of k: $k = 4$

(b) The set

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x + z = 2y\right\}$$

is a subspace of $\mathbb{R}^3$. Find a spanning set of S.

$S = \{(2y-z, y, z) : y, z \in \mathbb{R}\}$

$\quad = y(2,1,0) + z(-1,0,1)$

Spanning set of S: $\{(2,1,0), (-1,0,1)\}$

6. (15 pts) Determine whether the following sets of vectors are linear independent. Show all work; unjustified answers will not receive credit.

(a) $\{1, x, 2x, x^2\} \subset P_3(\mathbb{R})$

$0 \cdot 1 + 2 \cdot x + (-1)\,2x + 0 \cdot x^2 = 0$

There is a nontrivial linear combination equal to the zero vector.

Linearly independent?	**YES**	**NO**

(b) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \subset M_{2\times 2}(\mathbb{R})$

↳ Containing the zero vector of $M_{2\times2}(\mathbb{R})$

or $0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ same as above

Linearly independent?	**YES**	**NO**

(c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3$

$\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & 0 \end{bmatrix} = -(-1) - 1 = 0 \Rightarrow$ LD.

Or $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ same as above

Linearly independent?	**YES**	**NO**

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