MATH 165 Linear Algebra and Differential Equations

Midterm 1

October 4, 2022

Name: Solutions

UR ID: _____

Circle your instructor's name:

Salur Hopper Chaves Han

Instructions for the midterm you will be taking:

- The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden. If you have questions, get the attention of your exam proctor; otherwise no communication is allowed during the exam.
- Show your work! You may not receive full credit for a correct answer if insufficient work or insufficient justification is given. In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$. Be sure to include units when applicable!
- The exam is 1 hour, 15 minutes and is worth 100 points.

COPY THE HONOR PLEDGE AND SIGN (Cursive is not required)

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

_____

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YOUR SIGNATURE: _____

QUESTION	VALUE	SCORE
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
TOTAL	100	

1. (15 pts) Find the solution to the initial value problem

$$\frac{dy}{dx} = 2xy\ln(y), \quad y(0) = e^{-1}.$$

→ May assume $y > 0$ & $\ln(y) < 0$ (because $\ln(e^{-1}) = -1$)

$$\int \frac{dy}{y\ln(y)} = \int 2x\,dx + C$$

$$\ln|\ln(y)| = x^2 + C$$

$$\ln(-\ln(y)) = x^2 + C$$

$$\Rightarrow -\ln(y) = Ce^{x^2} \quad (C = e^C)$$

$x = 0, \; y = e^{-1}$

$$1 = -(-1) = Ce^0 = C$$

$$\Downarrow$$

$$\ln(y) = -e^{x^2} \rightarrow y = e^{-e^{x^2}}$$

Solution: $y = e^{-e^{x^2}}$

2. (15 pts) Find the general solution of the equation

$$\frac{dy}{dx} + 3y = x.$$

Hint: Integration by parts:

$$\int u\,dv = uv - \int v\,du.$$

Integrating factor: $I(x) = e^{\int 3\,dx} = e^{3x}$

$\rightarrow e^{3x}\frac{dy}{dx} + 3e^{3x}y = xe^{3x}$

$\frac{d}{dx}(e^{3x}y)$

$\rightarrow e^{3x}y = \int xe^{3x}\,dx = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

$u = x \quad v = \frac{1}{3}e^{3x}$

$du = dx \quad dv = e^{3x}dx$

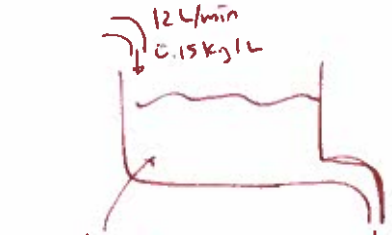
$\rightarrow y = \frac{1}{3}x - \frac{1}{9} + \frac{C}{e^{3x}}$

General solution:

$$y = \frac{1}{3}x - \frac{1}{9} + \frac{C}{e^{3x}}$$

3. (20 pts) Consider a tank containing 600 liters of salt water with an initial salt concentration of 0.03 kg/L. Every minute, 12 liters of a salt water solution with concentration 0.15 kg/L is added to the tank. Simultaneously, every minute, 12 liters of the well-mixed solution is drained from the tank.

(a) Let $y(t)$ denote the amount of salt in the tank (measured in kg) at time t. State an initial value problem whose solution is $y(t)$.



$\frac{dV}{dt} = 12 - 12 = 0 \Rightarrow V = 600\ (L)$

$\rightarrow \frac{dy}{dt} = 12 \cdot 0.15 - 12 \cdot \frac{y}{600} = 1.8 - \frac{y}{50}$

$\rightarrow y(0) = 18\ (kg)$

Initial value problem:

$$\frac{dy}{dt} = 1.8 - \frac{y}{50}, \quad y(0) = 18.$$

(b) Determine the amount of the salt in the tank $y(t)$ as $t \to \infty$.

$\frac{dy}{dt} + \frac{1}{50} y = 1.8$

I.F: $e^{\int \frac{1}{50} dt} = e^{t/50}$

$\frac{d}{dt}\left(e^{t/50} y\right) = 1.8 e^{t/50} \rightarrow e^{t/50} y = 90 e^{t/50} + C \Rightarrow y = 90 + C e^{-t/50}$

-72, from $y(0) = 18$

$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} 90 - 72 e^{-t/50} = 90.$

Amount of salt:

90 kg

4. (15 pts) Consider the following matrices:

$$A = \begin{bmatrix} -1 & -3 & 1 \\ 1 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -5 & 3 \\ 2 & -1 & -4 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Compute $2A - 3B$. If it is not defined, write "not defined" as the answer.

defined since A, B have the same size

$$2A = \begin{bmatrix} -2 & -6 & 2 \\ 2 & 0 & 6 \end{bmatrix}$$

$$3B = \begin{bmatrix} 3 & -15 & 9 \\ 6 & -3 & -12 \end{bmatrix}$$

Answer:

$$2A - 3B = \begin{bmatrix} -5 & 9 & -7 \\ -4 & 3 & 18 \end{bmatrix}$$

defined, 2×2

(b) Compute $AC + 5I_2$, where I_2 is the 2×2 identity matrix. If it is not defined, write "not defined" as the answer.

$$AC = \begin{bmatrix} -8 & -1 \\ -1 & 5 \end{bmatrix} \qquad 5I_2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

Answer:

$$AC + 5I_2 = \begin{bmatrix} -3 & -1 \\ -1 & 10 \end{bmatrix}$$

(c) Compute $C^T B$. If it is not defined, write "not defined" as the answer.

2×3 2×3

Not defined since the number of columns of C^T $(=3)$ is not equal to the number of rows of B $(=2)$

Answer: Not defined

5. (15 pts) Determine the values of x and y so the matrix

$$\begin{bmatrix} 0 & 1 & x-1 \\ -1 & 2x-y & -2 \\ y-5 & 2 & 0 \end{bmatrix}$$

is skew-symmetric.

$x-1 = -(y-5) = 5-y \Rightarrow x+y=6$

$2x-y=0 \Rightarrow y=2x$

$\Rightarrow x=2, \; y=4.$

$x =$ 2	$y =$ 4

6. (20 pts) Find the value(s) of a such that the following matrix has rank 2. Then find the value(s) of a such that the rank is 3.

$$A = \begin{bmatrix} 2 & 6 & 4 & -2 \\ -1 & 6 & -2 & -4 \\ 3 & 0 & a & 2 \end{bmatrix}$$

$\begin{bmatrix} 2 & 6 & 4 & -2 \\ -1 & 6 & -2 & -4 \\ 3 & 0 & a & 2 \end{bmatrix}$ P_{12} $\begin{bmatrix} -1 & 6 & -2 & -4 \\ 2 & 6 & 4 & -2 \\ 3 & 0 & a & 2 \end{bmatrix}$

$M_1(-1)$ $\begin{bmatrix} 1 & -6 & 2 & 4 \\ 2 & 6 & 4 & -2 \\ 3 & 0 & a & 2 \end{bmatrix}$

$A_{12}(-2)$ $\begin{bmatrix} 1 & -6 & 2 & 4 \\ 0 & 18 & 0 & 10 \\ 3 & 0 & a & 2 \end{bmatrix}$

$A_{13}(-3)$ $\begin{bmatrix} 1 & -6 & 2 & 4 \\ 0 & 18 & 0 & -10 \\ 0 & 18 & a-6 & -10 \end{bmatrix}$

$A_{23}(-1)$ $\begin{bmatrix} 1 & -6 & 2 & 4 \\ 0 & 18 & 0 & -10 \\ 0 & 0 & a-6 & 0 \end{bmatrix}$ $M_2(\frac{1}{18})$ $\begin{bmatrix} 1 & -6 & 2 & 4 \\ 0 & 1 & 0 & -5/9 \\ 0 & 0 & a-6 & 0 \end{bmatrix}$

If $a-6=0 \Rightarrow \text{rank}(A)=2$

If $a-6 \neq 0 \Rightarrow \text{rank}(A)=3$ (after $M_3(\frac{1}{a-6})$, there are 3 leading 1's)

Value(s) of a such that $\text{rank}(A) = 2$:	**Value(s) of a such that** $\text{rank}(A) = 3$:
6	any $a \neq 6$.

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