1. (20 pts) Determine whether each given set S is a subspace of the given vector space V. If so, give a proof; if not, state a property it fails to satisfy.

(a) $V = \mathbb{R}^3$, and $S = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 = 3a_2 \text{ and } a_3 = -a_2\}$.

$$\Updownarrow$$

$$\text{Nullspace}\left(\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right)$$

Circle final answer. S is a subspace: YES or NO

(b) $V = \mathbb{R}^3$, and $S = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$. } has zero vector and closed under scaling

but not closed under +

$(0, \sqrt{2}, 1)$ in S

$(\frac{\sqrt{3}}{\sqrt{5}}, 1, 0)$ in S

but sum $(\sqrt{\tfrac{3}{5}}, 1+\sqrt{2}, 1)$ not in S

as $5(\sqrt{\tfrac{3}{5}})^2 - 3(1+\sqrt{2})^2 + 6(1^2)$

$= 3 - 3(1+\sqrt{2})^2 + 6$

$= \cancel{3} - \cancel{3} - 6\sqrt{2} - \cancel{6} + \cancel{6} \neq 0$

Circle final answer. S is a subspace: YES or NO

(c) $V = M_{2\times 2}(\mathbb{R})$, and $S = \left\{ A \in M_{2\times 2}(\mathbb{R}) \;\middle|\; \mathrm{tr}(A) = 0 \right\}$.

$\parallel$

See HW

Circle final answer. S is a subspace: YES or NO

(d) $V = P_3(\mathbb{R})$, and $S = \left\{ p(x) \in P_3(\mathbb{R}) \;\middle|\; p'(1) = p(0) \right\}$.

$$\parallel$$

$$\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_1 + 2a_2 + 3a_3 = a_0 \}$$

$$\parallel\wr$$

$$\left\{ \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \;\middle|\; a_0 - a_1 - 2a_2 - 3a_3 = 0 \right\}$$

$$\parallel$$

$$\text{Nullspace}([1 \ -1 \ -2 \ -3])$$

Circle final answer. S is a subspace: YES or NO

2. (16 pts)

(a) Use any method to find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 0 & 5 & -4 \end{bmatrix}.$$

Include all details.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[A_{12}(-1)]{} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow A_{23}(-5),\ A_{21}(-2)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 3 & -2 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 5 & -5 & 1 \end{array}\right]$$

$$\downarrow A_{32}(1),\ A_{31}(-5)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -22 & 23 & -5 \\ 0 & 1 & 0 & 4 & -4 & 1 \\ 0 & 0 & 1 & 5 & -5 & 1 \end{array}\right]$$

$$\begin{bmatrix} -22 & 23 & -5 \\ 4 & -4 & 1 \\ 5 & -5 & 1 \end{bmatrix} = A^{-1}$$

Inverse:	See above

(b) Let $\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, $\vec{a}_3 = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Determine if $\vec{b} \in \operatorname{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ or not. If so, write $\vec{b}$ explicitly as a linear combination of $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$; if not, explain why not.

$\mathbb{A}$ same as in part (a)

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 1 \\ 0 & 5 & -4 & 1 \end{array}\right]}_{\mathbb{A}} \iff \mathbb{A}\vec{x} = \vec{b}$$

$$\sim \left[\begin{array}{cc} 1 & 2 \end{array}\right.$$

$$\Updownarrow$$

$$\vec{x} = \mathbb{A}^{-1}\vec{b}$$

$$= \begin{bmatrix} -22 & 23 & -5 \\ 4 & -4 & 1 \\ 5 & -5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 18 \\ -3 \\ -4 \end{bmatrix}$$

$$\therefore \quad \vec{b} = 18\vec{a}_1 - 3\vec{a}_2 - 4\vec{a}_3$$

Circle answer. $\vec{b} \in \operatorname{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$: **YES** or **NO**
If answered YES, write $\vec{b}$ as linear combination here: $\vec{b} = 18\vec{a}_1 - 3\vec{a}_2 - 4\vec{a}_3$

3. (14 pts) Show that $\{3x^2 + x + 1, 2x + 1, 2\}$ is a basis for $P_2(\mathbb{R})$.

$$a + bx + cx^2 \Longleftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Use isomorphism $P_2(\mathbb{R}) \cong \mathbb{R}^3$

So equivalent to show $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^3$

Basis $\Longleftrightarrow \det\left(\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \right) \neq 0$

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{vmatrix} \underset{\text{3rd row}}{\overset{\text{cofactor}}{=}} (+)3 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = 3(0-4) = -12 \neq 0$$

So is a basis!

4. (19 pts)

(a) Compute the determinant of the matrix A, defined by

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \underset{\text{4th row}}{\overset{\text{cofactor}}{=}} (-1)\cdot 1 \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} \overset{\text{cofactor 2nd row}}{=} (-1)(+1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = (-1)(-1) = +1$$

det(A) = 1

(b) Answer the following questions regarding matrix A from part (a).

Circle answer. A **is invertible: YES or NO**
Explanation: det $= 1 \neq 0$

(c) Suppose B is a 3×3 matrix with determinant 2. C is obtained from B by interchanging two rows. D is obtained from B by adding 5 times row 1 to row 2 (while leaving row 1 unchanged). Find the following determinants (no work needed - just put the final answers alone in the answer boxes. There is space for optional work below the answer boxes.)

$\det(\mathbf{B}^3) = (\det B)^3 = 2^3 = 8$	$\det(\mathbf{B}^T) = \det B = 2$
$\det(\mathbf{B}^{-1}) = \frac{1}{\det B} = \frac{1}{2}$	$\det(\mathbf{2B}) = 2^3 \det(B) = 2^4 = 16$
$\det(\mathbf{C}) = -\det B = -2$	$\det(\mathbf{D}) = \det B = 2$

(d) Suppose K is a $m \times n$ matrix of nullity 5 and where Col(K), the column space of K, has a basis given by the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 7 \end{bmatrix}$ $\Big\}\; m=4$. State what m and n have to be. (No work needed though there is space for optional work below the answer boxes)

m = 4	n = 7

$$n = \text{rank} + \text{nullity} = \underbrace{2}_{\text{basis for col space}} + 5 = 7$$

5. (16 pts) The matrix

$$A = \begin{pmatrix} 1 & -2 & 1 & 3 \\ 3 & -6 & 2 & 7 \\ 4 & -8 & 3 & 10 \end{pmatrix}$$

has reduced row echelon form given by

bdd

$$U = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

(a) Find a basis for the nullspace of A.

$$\text{Nullspace}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$= \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right)$$

Basis for nullspace: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

(b) Find a basis for the column space of A.

Use pivot cols from original A

Basis for column space: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

(c) Compute the rank and nullity of A.

Rank of A is: 2

Nullity of A is: 2

6. (15 pts) Choose the correct answer (which should be universally correct) out of the list provided. You do not need to show work, and partial credit will not be offered.

(a) Let $S := \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors in $\mathbb{R}^n$ for $n \geq 2$. Suppose A is the $n \times n$ matrix given by $A := [\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_n]$, where $\det(A) = 0$. Then $\vec{v}_1$ can always be written as a linear combination of $\vec{v}_2, \ldots, \vec{v}_n$.

$\det A = 0 \to \{\vec{v}_1, \ldots, \vec{v}_n\}$ LD but need not imply $\vec{v}_1 =$ comb. of others

Expl: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ has $\det A = 0$ but $\vec{v}_1 \neq c_2\vec{v}_2 + c_3\vec{v}_3$ for any c_2, c_3

☐ Statement (a) is true.

☒ Statement (a) is false.

(b) Let $A \in M_{m\times n}(\mathbb{R})$ and suppose $1 \leq m < n$. Then:

☐ $\dim(nullspace(A)) \leq n - m$. $\quad$ $\underset{\text{rank}}{(\#\text{pivots})} \leq m$

☐ $\dim(nullspace(A)) = n - m$. $\quad$ So rank + nullity $= n$

☒ $\dim(nullspace(A)) \geq n - m$. $\quad$ $\to$ nullity $\geq n - m$

☐ None of the above conclusions can be drawn without more information.

(c) If A is a lower-triangular 2×2 matrix with entries in $\mathbb{R}$ and $tr(A) = 0$, then:

$A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ is lower-Δ $\quad tr A = 0 \to c = -a$

so $A = \begin{bmatrix} a & 0 \\ b & -a \end{bmatrix} \to \det(A) = -a^2 \leq 0$

☐ $det(A) \geq 0$.

☐ $det(A) = 0$.

☒ $det(A) \leq 0$.

☐ None of the above conclusions can be drawn without more information.

(d) Let $S := \{p(x) \in P_3(\mathbb{R}) \mid p(x) = p(-x)\}$. Then:

$S = \{a + bx + cx^2 + dx^3 \mid b = d = 0\}$
$= \{a + cx^2\} = \underbrace{\text{Span}(1, x^2)}_{\text{2-D. Subspace}}$

☐ S is not a subspace of $P_3(\mathbb{R})$.

☐ S is a subspace of $P_3(\mathbb{R})$ with dimension 4.

☐ S is a subspace of $P_3(\mathbb{R})$ with dimension 3.

☒ S is a subspace of $P_3(\mathbb{R})$ with dimension 2.

☐ S is a subspace of $P_3(\mathbb{R})$ with dimension 1.

(e) If A is an $n \times n$ invertible matrix, then $A + A^T$ is also invertible.

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\det A = 1 \neq 0$ so invertible

but $A + A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \to$ <u>not</u> invertible

☐ True

☒ False