Math 165

Midterm 1

Oct 3, 2023

Name: ______________________________

UR ID: ______________________________

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I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

1. **(18 pts)** Find the solution to the initial value problem

$$4x + y^2 e^{-x}\frac{dy}{dx} = 0, \quad y(0) = 3.$$

$$y^2 e^{-x}\frac{dy}{dx} = -4x$$

$$\int y^2\,dy = -\int 4xe^x\,dx + C$$

$$\frac{y^3}{3} = -4\left(\int \overset{u}{x}\,\overset{dv}{e^x dx}\right) + C$$

$$= uv - \int v\,du$$

$$= -4\left(xe^x - e^x\right) + C$$

$$\frac{y^3}{3} = -4xe^x + 4e^x + C$$

$$\boxed{y(0)=3: \quad 9 = 4 + C \rightarrow C = 5}$$

$$\therefore \frac{y^3}{3} = -4xe^x + 4e^x + 5$$

$$y^3 = -12xe^x + 12e^x + 15$$

$$y = \sqrt[3]{-12xe^x + 12e^x + 15}$$

Answer: $y = \sqrt[3]{-12xe^x + 12e^x + 15}$

2. **(18 pts)** Find the general solution of the differential equation (for $x > -1$)

$$\frac{dy}{dx} - \frac{3y}{x+1} = (x+1)^4.$$

$$P(x) = \frac{-3}{x+1}$$

$$\text{Int. Factor} = \psi = e^{\int P(x)dx} = e^{-3\ln(x+1)} = e^{\ln(x+1)^{-3}} = (x+1)^{-3}$$

$$\therefore \quad \frac{d}{dx}\left(y(x+1)^{-3}\right) = (x+1)^4 (x+1)^{-3} = x+1$$

$$\therefore \quad y(x+1)^{-3} = \frac{(x+1)^2}{2} + C$$

$$\left(\text{or } \frac{x^2}{2} + x + C' \text{ also ok}\right)$$

$$y = \frac{(x+1)^5}{2} + C(x+1)^3$$

$$\left(\text{Also correct is } y = \left(\frac{x^2}{2}+x\right)(x+1)^3 + D(x+1)^3\right)$$

Answer: $y(x) = \dfrac{(x+1)^5}{5} + C(x+1)^3$ OR $\left(\frac{x^2}{2}+x+D\right)(x+1)^3$

also OK

3. (12 pts) Find all values of r such that $y = e^{rx}$ is a solution to the differential equation

$$y''' - 4y' = 0.$$

$r^3 e^{rx} - 4re^{rx} = 0$

$\rightarrow \quad r^3 - 4r = 0$

$r(r^2 - 4) = 0$

$r(r-2)(r+2) = 0$

The r values that work are: $-2, 2, 0$

4. (14 pts) A tank initially contains 30 L of water in which there is 20 g of salt dissolved. A solution containing 2 g/L of salt is pumped into the container at a rate of 3 L/min, and the well-stirred mixture runs out at a rate of 2 L/min.

(a) Derive an initial-value problem describing $s(t)$, the amount of salt in the tank at time t. Your differential equation should only contain two variables s and t (it is understood that your differential equation only holds until the tank begins to overflow) and you should state the relevant initial condition(s) also.

$$\frac{dV}{dt} = 3 - 2 = 1 \longrightarrow V = t + C \xrightarrow{V_0 = 30} \boxed{V = t + 30}$$

$$\frac{ds}{dt} = 3 \cdot (2) - 2\left(\frac{s}{V}\right) = 6 - \frac{2s}{t+30} \qquad s(0) = 20$$

Answer: $\frac{ds}{dt} = 6 - \frac{2s}{t+30}$ $\qquad s(0) = 20$ grams

(b) Determine the amount of salt in the tank after 30 minutes. (Assume the tank is large enough that overflow does not occur within the first 30 minutes).

$$\frac{ds}{dt} + \left(\frac{2}{t+30}\right)s = 6 \qquad \mu = e^{\int \frac{2}{t+30}dt} = e^{2\ln(t+30)} = (t+30)^2$$

$$\therefore \frac{d}{dt}\left(s(t+30)^2\right) = 6(t+30)^2 \longrightarrow s(t+30)^2 = \frac{6(t+30)^3}{3} + C$$

$$s = 2(t+30) + \frac{C}{(t+30)^2}$$

$s(0) = 20$

$$\left(20 = 60 + \frac{C}{(30)^2} \rightarrow -40(900) = C \rightarrow C = -36000\right.$$

set $t = 30$

Answer: $2(60) - \frac{36000}{(3600)} = \boxed{110 \text{ grams}}$

5. (18 pts)

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}.$$

Compute each of the following or justify why the expression is not well defined. If the expression is not defined put NOT DEFINED in the solution box, otherwise enter your final answer there. Show your work.

(a) $(A + A^T)^2$ $= \begin{bmatrix} -4 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -4 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 18 & -3 & -3 \\ -3 & 2 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

Solutions (if any): $\begin{bmatrix} 18 & -3 & -3 \\ -3 & 2 & 1 \\ -3 & 1 & 2 \end{bmatrix}$

(b) $(A + BC)^T$

$\underbrace{A}_{3\times 3} + \underbrace{BC}_{1\times \cancel{3}\, \cancel{3}\times 1 = 1\times 1}$

Can't add different shape matrices

Solutions (if any): NOT DEFINED

(c) $B^T C^T - 2A$

$$= \underbrace{\underset{3\times 1}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 4 & -2 & 6 \end{bmatrix}}}_{3\times 3} - 2\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & 6 \\ 0 & 0 & 0 \\ 8 & -4 & 12 \end{bmatrix} - \begin{bmatrix} -4 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & -2 & 4 \\ -2 & 0 & 0 \\ 8 & -6 & 12 \end{bmatrix}$$

Solutions (if any): $\begin{bmatrix} 8 & -2 & 4 \\ -2 & 0 & 0 \\ 8 & -6 & 12 \end{bmatrix}$

(d) $(CB + BC)^2$

$$\underbrace{\underbrace{CB}_{3\times 1\ 1\times 3}}_{3\times 3} \qquad \underbrace{\underbrace{BC}_{1\times 3\ 3\times 1}}_{1\times 1}$$

can't add different size

Solutions (if any): NOT DEFINED

6. (20 pts) For parts (a) and (b), use Gaussian (or Gauss-Jordan) elimination to find all solutions to the given system or show that no solutions exist. Show all steps.

(a)

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 10 \\ x_1 - x_2 + x_3 &= -1 \\ 3x_1 + x_2 + 2x_3 &= 8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 10 \\ 1 & -1 & 1 & -1 \\ 3 & 1 & 2 & 8 \end{array}\right] \xrightarrow[A_{23}(-3)]{A_{21}(-2)} \left[\begin{array}{ccc|c} 0 & 5 & -3 & 12 \\ 1 & -1 & 1 & -1 \\ 0 & 4 & -1 & 11 \end{array}\right]$$

$$\xrightarrow{A_{31}(-1)} \left[\begin{array}{ccc|c} 0 & 1 & -2 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 4 & -1 & 11 \end{array}\right] \xrightarrow[A_{13}(-4)]{A_{12}(1)} \left[\begin{array}{ccc|c} 0 & 1 & -2 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 7 & 7 \end{array}\right] \xrightarrow{M_3(\frac{1}{7})} \left[\begin{array}{ccc|c} 0 & 1 & -2 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[A_{31}(2)]{A_{32}(1)} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Solutions (if any): $x_1 = 1$, $x_2 = 3$, $x_3 = 1$ Unique Solution.

(b)

$$\begin{aligned} x_1 + x_2 - x_3 &= 5 \\ x_1 + x_3 &= 2 \\ 4x_1 + x_2 + 2x_3 &= 12 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 1 & 0 & 1 & 2 \\ 4 & 1 & 2 & 12 \end{array}\right] \xrightarrow[A_{13}(-4)]{A_{12}(-1)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & -1 & 2 & -3 \\ 0 & -3 & 6 & -8 \end{array}\right]$$

$$\xrightarrow{M_2(-1)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -2 & 3 \\ 0 & -3 & 6 & -8 \end{array}\right] \xrightarrow[A_{23}(3)]{A_{21}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$\xrightarrow{M_3(-\frac{1}{2})}$ ← not really needed

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

"$0=1$" no solution

Solutions (if any): NO SOLUTION.

EXTRA PAGE. You may use this page if you run out of space. Be sure to label your problems on this page and also include a note on the original page telling the graders to look for your work here.