Math 165

Final

Dec 17, 2023

Name: ______________________________

Student ID: ______________________________

PLEASE COPY THE HONOR PLEDGE AND SIGN:
(Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:____________________________________

Part A		
QUESTION	VALUE	SCORE
1	16	
2	20	
3	14	
4	15	
5	20	
6	15	
TOTAL	100	

Part B		
QUESTION	VALUE	SCORE
1	17	
2	16	
3	17	
4	15	
5	18	
6	17	
TOTAL	100	

Part A

1. (16 pts) Determine whether each given set S is a subspace of the given vector space V. **If so, give a proof; if not, state a property it fails to satisfy.**

(a)(4 points) $V = \mathbb{R}^3$, and $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 3x = 2y + 5z \right\}$.

$$= \text{Nullspace}\left(\begin{array}{ccc} x & y & z \\ \end{array} \left[3, -2, -5\right]\right)$$

Circle final answer. S is a subspace: YES or NO?

(b)(4 points) $V = \mathbb{R}^2$, and $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x + y = xy \right\}$.

When $x = y$: $2y = y^2 \to y = 0, y = 2$

So $(2,2) \in S$

but $(2,2) + (2,2) = (4,4) \notin S$

Not closed under $+$

Circle final answer. S is a subspace: YES or NO?

(c)(4 points) $V = M_3(\mathbb{R})$, and $S = \left\{ A \in M_3(\mathbb{R}) \;\middle|\; \det A = 0 \right\}$.

See solutions in HW

or $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$

$\det\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \neq 0$

Circle final answer. S is a subspace: YES or NO?

(d)(4 points) $V = P_3(\mathbb{R})$, and $S = \left\{ f \in P_3(\mathbb{R}) \;\middle|\; f(2)^2 - f(x^2) = f(x^3) \right\}$.

due to this square it is nonlinear

$f(x) = a + bx + cx^2 + dx^3$

Conditions: $\underbrace{(a+2b+4c+8d)^2}_{``f(2)"} - \underbrace{(a+bx^2+cx^4+dx^6)}_{``f(x^2)"} = \underbrace{a+bx^3+cx^6+dx^9}_{``f(x^3)"}$

which needs $\left\{ \begin{array}{ll} (a+2b+4c+8d)^2 - a = a & \\ b = 0 & (\text{no } x^2 \text{ on right}) \\ c = 0 & (\text{no } x^4 \text{ on right}) \\ d = c & (\text{compare } x^6 \text{ on left + right}) \end{array} \right.$

so need $a^2 - a = a$, $b = c = d = 0$ needed

$a^2 = 2a$, $b = c = d = 0$

$\rightarrow a = 0, 2$, $b = c = d = 0$ $\quad S = \{0, 2\} \leftarrow$ two things in S

not a subspace.

Circle final answer. S is a subspace: YES or NO?

2. (20 pts)

[10 points] (a) Find the solution to the differential equation

$$(y + x^2 y)\frac{dy}{dx} = 4$$

which satisfies the initial condition $y(0) = 2$.

$$y(1+x^2)\frac{dy}{dx} = 4$$

$$\text{"}\ y\,dy = \frac{4\,dx}{1+x^2}\ \text{"}$$

$$\int y\,dy = \int \frac{4\,dx}{1+x^2} + C$$

$$\frac{y^2}{2} = 4\arctan(x) + C$$

$$y^2 = 8\arctan(x) + D \qquad (D = 2C)$$

$$y = \pm\sqrt{8\arctan(x) + D}$$

$y(0)=2$: $\quad 2 = \pm\sqrt{8\arctan(0) + D}$ $\longrightarrow$ (Use + sign and $2 = \sqrt{D}$ $\longrightarrow D = 4$)

$$\therefore\quad y = \sqrt{8\arctan(x) + 4}$$

Answer: $y(x) =$ $\sqrt{8\arctan(x) + 4}$

[10 points] (b) Find the solution to the differential equation

$$x\frac{dy}{dx} + 2y - 4x^2 = 0$$

which satisfies the initial condition $y(2) = 6$.

$$\frac{dy}{dx} + \overbrace{\frac{2}{x}}^{=P} y = \frac{4x^2}{x} = \overbrace{4x}^{=Q}$$

$$\psi = \text{Integration factor} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$\therefore \quad \frac{d}{dx}(y\psi) = 4x\,(x^2) = 4x^3$$

$$\longrightarrow \quad y\psi = \int 4x^3 dx + C = x^4 + C$$

$$\longrightarrow \quad y = \frac{x^4 + C}{x^2} = x^2 + \frac{C}{x^2}$$

$$y(2) = 6: \quad 6 = 2^2 + \frac{C}{2^2} \longrightarrow \frac{C}{4} = 2$$

$$\longrightarrow C = 8$$

$$\therefore \quad y = x^2 + \frac{8}{x^2}$$

Answer: $y(x) = x^2 + \frac{8}{x^2}$

3. (14 pts)

Use Gauss-Jordan row reduction to find the inverse of the matrix

$$A = \begin{bmatrix} 5 & 6 & 6 \\ 2 & 2 & 2 \\ 6 & 6 & 2 \end{bmatrix}$$

if it exists.

$[A \mid I]$ =

$$\left[\begin{array}{ccc|ccc} 5 & 6 & 6 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 6 & 6 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[A_{21}(-2)]{A_{23}(-3)} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & -2 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{A_{12}(-2)} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & -2 & 0 \\ 0 & -2 & -2 & -2 & 5 & 0 \\ 0 & 0 & -4 & 0 & -3 & 1 \end{array}\right]$$

$$\xrightarrow[\text{then } M_2(-\frac{1}{2}),\ M_3(-\frac{1}{4})]{A_{21}(1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & 0 \\ 0 & 1 & 1 & 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{3}{4} & -\frac{1}{4} \end{array}\right]$$

$$\xrightarrow{A_{32}(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & 0 \\ 0 & 1 & 0 & 1 & -\frac{13}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{3}{4} & -\frac{1}{4} \end{array}\right]$$

Answer: $A^{-1} = \begin{bmatrix} -1 & 3 & 0 \\ 1 & -\frac{13}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$

4. (15 pts) Consider a system of linear equations $A\mathbf{x} = \mathbf{b}$ where A is an $m \times n$ matrix, where m and $\boxed{n \text{ are positive integers.}}$ Let $r = rank(A)$ and $z = nullity(A)$. In each of the following cases, what can be said about the number of solutions to the system? **(Mark only one of the choices in each part.)**

rank-nullity thm
$r + z = n$

1. If $r = z$ and $\mathbf{b}$ is the zero vector, then the system

   ☐ is inconsistent.

   ☐ has a unique solution.

   ☒ has infinitely many solutions.

   ☐ Further information is necessary to determine an answer.

   $r = z \to r = z = \frac{n}{2} > 0$ as $n > 0$
   So $z > 0 \to$ free variables exist
   $\to r > 0 \to$ pivots + bound variables also exist
   as $Ax = 0$ always has $x = 0$ soln. $\to \infty$ solns

2. If $r = z$ and $\mathbf{b}$ is not the zero vector, then the system

   ☐ is inconsistent.

   ☒ does not have a unique solution.

   ☐ has infinitely many solutions.

   ☒ Further information is necessary to determine an answer.

   Either is OK

   Still $r = z$
   $Ax = b$ reduces to $Ux = d$ RREF
   ~~If some U has~~
   As free var exist can't be unique soln but might be $[0 \ldots 0 | d_n \neq 0]$ no solutions or ∞ soln case
   Both B & D are reasonable answers + we give credit for both

3. If $z = 0$, $n < m$ and $\mathbf{b} \neq \mathbf{0}$ then the system

   ☐ is inconsistent.

   ☐ has a unique solution.

   ☐ has infinitely many solutions.

   ☒ Further information is necessary to determine an answer.

   $z = 0 \to$ no free vars so can't be ∞ many
   $r + z = n \to r = n$ but $n < m$ so some b not in colspace(A)
   $Ax = b$ can have no soln or ~~∞ many~~ unique soln depending if $b \in Col(A)$

4. If $z \neq 0$ and $\mathbf{b} = \mathbf{0}$, then the system

   ☐ is inconsistent.

   ☐ has a unique solution.

   ☒ has infinitely many solutions.

   ☐ Further information is necessary to determine an answer.

   $z \neq 0 \to$ Free var exist
   $\to$ Can't be unique soln
   As $Ax = 0$ always has soln $x = 0$
   $\to \infty$ many solns

5. If $m > n$, $z = n$ and $\mathbf{b} \neq \mathbf{0}$, then the system

   ☒ is inconsistent.

   ☐ has a unique solution.

   ☐ has infinitely many solutions.

   ☐ Further information is necessary to determine an answer.

   $z = n \to$ all variables free
   $\to A = 0$ = zero matrix (no pivots)
   $\therefore 0x = b \neq 0$ has no solutions

5. (20 pts)

[10 points] (a) Find the determinant of

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & c \\ 1 & 4 & c^2 \end{pmatrix}$$

as a function of c. For which value(s) of c is M not invertible?

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & c \\ 1 & 4 & c^2 \end{vmatrix} \xrightarrow[A_{13}(-1)]{A_{12}(-1)} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & c-1 \\ 0 & 3 & c^2-1 \end{vmatrix} \xrightarrow[\text{1st Col}]{\text{Cofactor}} 1 \cdot \begin{vmatrix} 1 & c-1 \\ 3 & c^2-1 \end{vmatrix}$$

$$= (c^2-1)-3(c-1)$$

$$= c^2-3c+2$$

$$= (c-2)(c-1)$$

Answer: $\det(M) =$ $(c-1)(c-2)$

Value(s) of c when M is not invertible are: $1, 2$

[10 points] (b) Suppose A is a 4×4 matrix with $\det(A) = -2$ and B is obtained from A by subtracting 2 times row 3 from row 2. Then:

(i) **Answer:** $\det(2A) = 2^4 \det(A)$
$= -32$

(ii) **Answer:** $\det(A^T) = \det(A)$
$= -2$

(iii) **Answer:** $\det(A^{-1}) =$ $1/\det A$
$= -1/2$

(iv) **Answer:** $\det(A^3) = (\det A)^3$
$= (-2)^3 = -8$

(v) **Answer:** $\det(B) = \det(A)$
$= -2$

6. (15 pts)

The matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 4 & -1 & 1 & -1 \\ 8 & -2 & 3 & -1 \end{pmatrix}$$

is row-equivalent to the matrix

$$B = \begin{pmatrix} 4 & -1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{not fully reduced}$$

$$\xrightarrow[\substack{A_{31}(-4) \\ A_{32}(-2) \\ \text{then } A_{21}(-3)}]{\text{reduce}} \begin{pmatrix} 4 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{M_1(\frac{1}{4})} \boxed{U = \begin{bmatrix} 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{RREF}}$$

[3 points] (a) The rank of A is

Answer: Rank of A is: 3

[3 points] (b) The nullity of A is

Answer: Nullity of A is: 1 ← rank+nullity = 4

[3 points] (c) List a set of basis vectors for the column space of A.

Answer: Basis for column space of A is:

$$\begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

[3 points] (d) List a set of basis vectors for the null space of A. $\rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = \text{Span}\left(\begin{matrix} \frac{1}{4} \\ 1 \\ 0 \\ 0 \end{matrix}\right) = \text{Span}\left(\begin{matrix} 1 \\ 4 \\ 0 \\ 0 \end{matrix}\right)$

Answer: Basis for nullspace of A is:

$$\begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

[3 points] (e) Give an example of a nontrivial linear dependency amongst the columns of A.

Answer: Example of nontrivial linear dependency is:

$$\vec{V_1} = -4\vec{V_2}$$

Part B

1. (17 pts) For the differential equation

$$(D^2+3)^2(D+1)^2y = e^t,$$

[7 points] (a) Find the general solution y_c to its associated homogeneous differential equation $(D^2+3)^2(D+1)^2y = 0$.

roots -1 $\quad -\sqrt{3}i, +\sqrt{3}i$

Mult $\quad 2 \quad 2 \quad 2$

$$y_c = C_1e^{-t}+C_2te^{-t}+\underbrace{C_3e^{-\sqrt{3}it}+C_4e^{+\sqrt{3}it}}+\underbrace{C_5te^{-\sqrt{3}it}+C_6te^{+\sqrt{3}it}}$$

$D_3\cos(\sqrt{3}t)+D_4\sin(\sqrt{3}t)+D_5t\cos(\sqrt{3}t)+D_6t\sin(\sqrt{3}t)$

Answer: $y_c =$ $C_1e^{-t}+C_2te^{-t}+D_3\cos(\sqrt{3}t)+D_4\sin(\sqrt{3}t)+D_5t\cos(\sqrt{3}t)+D_6\sin(\sqrt{3}t)$

[7 points](b) Find a particular solution y_p to the differential equation.

$y_p = Ae^t$ note $Dy = 1\cdot y$ so plug in

to $(D^2+3)^2(D+1)^2y = e^t$

get $(1^2+3)^2(1+1)^2(Ae^t) = e^t$

$64Ae^t = e^t$

$A = 1/64$

$\therefore$ $y_p = \frac{e^t}{64}$

Answer: $y_p =$ $\frac{1}{64}e^t$

[3 points](c) Determine the general solution to the differential equation.

Answer: $y =$ Add (a) & (b)

2. (16 pts) Consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -3 & 2 & -1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

[4 points] (a) Determine the eigenvalues of A.

Lower-Δ so diagonals entries

Answer: Eigenvalues are: $1, -1$

[8 points] (b) Determine the eigenspaces corresponding to each of the eigenvalues of A. In your answer make sure to label so that it can be determined which eigenspace belongs to which eigenvalue.

1-eigenspace $= \text{Nullspace} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -3 & 2 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \text{Nullspace} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} = \text{Nullspace} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$

$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -\frac{1}{2}x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{Span}\left(\begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$

(-1)-eigenspace $= \text{Nullspace} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix} = \text{Nullspace} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \text{Nullspace} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2x_4 \\ x_4 \end{bmatrix} \right\} = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right)$

Answer: Eigenspaces are:

1-eigenspace $= \text{Span}\left(\begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \leftarrow 2D$

-1-eigenspace $= \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right) \leftarrow 1D$

$2+1=3<4$

[4 points] (c) Determine if A is defective or nondefective. Justify your answer.

Circle final answer. A is defective or nondefective?

Explanation: At most 3 LI eigenvectors for 4x4 matrix → Not enough!

3. (17 pts) Solve the initial value problem

$$y'' - 2y' + 5y = 0$$

with $y(0) = -3, y'(0) = 1$.

$(D^2 - 2D + 5)y = 0$

roots $= \frac{2 \pm \sqrt{2^2 - 4(5)}}{2}$

$= 1 \pm \sqrt{-4}$

$= 1 \pm 2i$

$y = C_1 e^t \cos(2t) + C_2 e^t \sin(2t)$

$y(0) = -3$: $\boxed{-3 = C_1}$

$y' = C_1 e^t \cos(2t) - C_1 e^t 2\sin(2t) + C_2 e^t \sin(2t) + 2C_2 e^t \cos(2t)$

$y'(0) = 1$: $1 = C_1 + 2C_2$

$1 = -3 + 2C_2 \rightarrow C_2 = 2$

Answer: $y(x) = -3e^t \cos(2t) + 2e^t \sin(2t)$

4. (15 pts) A small mass m is attached to a wall with a horizontal spring with spring constant k. The floor the system lies on has friction coefficient c. The y-axis is perpendicular to the wall, pointing away from it, and the mass is confined to move along only this direction for this problem. As usual, we set $y = 0$ to be the rest position of the spring. Under these assumptions, with no further forces besides those of the spring and friction, the spring displacement y satisfies the following "simple harmonic oscillator" differential equation:

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0. \qquad \left(\underbrace{D^2 + \frac{c}{m}D + \frac{k}{m}}\right)y = 0$$

where $m, k, c > 0$ and the independent variable is time t. Each part in the following scenarios is independent of each other part with different parameters. In each part either select the most correct answer out of the selection given or enter a numerical answer if required.

roots $= -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \leftarrow$ roots $= \dfrac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2}$

(a) Suppose that in suitable units, the values of c, k, m are $c = 3, k = 1, m = 1$. If the spring is displaced from rest the following will occur: $c^2 - 4km = 9 - 4 = 5 > 0$ two real roots

- ☒ The spring will return to rest over time without undergoing oscillations. ✓
- ☐ The spring will undergo decaying oscillations, whose amplitude decays exponentially over time.
- ☐ The spring will undergo osclliations, whose amplitude remains constant over time.

(b) Suppose that in suitable units, the values of c, k, m are $c = 4, k = 3, m = 2$. If the spring is displaced from rest the following will occur: $c^2 - 4km = 16 - 4(6) = -8 < 0$

complex roots with $-\frac{4}{2(2)} \pm \frac{1}{4}\sqrt{-8}$ w/ real part (−) decay osc

- ☐ The spring will return to rest over time without undergoing oscillations.
- ☒ The spring will undergo decaying oscillations, whose amplitude decays exponentially over time.
- ☐ The spring will undergo osclliations, whose amplitude remains constant over time.

(c) Suppose that in suitable units, the values of c, k, m are $c = 0, k = 2, m = 2$. The natural frequency of the system is equal to:

$c^2 - 4km = 0^2 - 4(4)$

roots $= 0 \pm \frac{1}{2(2)}\sqrt{-4(4)} = \pm i$

Natural Frequency= 1 (circular frequency) OR $\frac{1}{2\pi}$ (freq)

$y = C_1\cos(1t) + C_2\sin(1t)$

(d) Suppose that in suitable units, the values of c, k, m are $c = 0, k = 2, m = 2$ and a motor drives the spring with force $F = 10\cos(\omega t)$. For which value of ω will the response of the system be strongest in terms of magnitude of oscillations?

Value of ω for strongest response: 1 (same as (c))

5. (18 pts)

[9 points] (a) Suppose a system $\hat{x}' = \mathbb{A}\hat{x}$ where $\mathbb{A}$ is a 2×2 matrix has general solution

$$\hat{x} = C_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find $\mathbb{A}$.

3, 5 eigenvalues

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ eigenvectors

$$A = S\Lambda S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\left(\begin{bmatrix} 6 & 5 \\ 3 & 10 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 & 4 \\ -4 & 17 \end{bmatrix}$$

Answer: $\mathbb{A} =$ $\begin{bmatrix} \frac{7}{3} & \frac{4}{3} \\ -\frac{4}{3} & \frac{17}{3} \end{bmatrix}$

[9 points] (b) Let $\mathbb{B}$ be a 2×2 real matrix which has eigenvalue $2 + 3i$ with corresponding eigenvector $\begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$. Write down the general solution to $\hat{x}' = \mathbb{B}\hat{x}$ where the independent variable is time t. **Please make sure that the two basis solutions used in the final form of your general solution are real valued quantities.**

$$e^{(2+3i)t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

$$= \left(e^{2t}\cos(3t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{2t}\sin(3t)\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) + i\left(e^{2t}\cos(3t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + e^{2t}\sin(3t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{2t}\cos(3t) \\ e^{2t}\cos(3t) - 2e^{2t}\sin(3t) \end{bmatrix} + i\begin{bmatrix} e^{2t}\sin(3t) \\ 2e^{2t}\cos(3t) + e^{2t}\sin(3t) \end{bmatrix}$$

Answer: $\hat{x}(t) =$ $C_1\begin{bmatrix} e^{2t}\cos(3t) \\ e^{2t}\cos(3t) - 2e^{2t}\sin(3t) \end{bmatrix} + C_2\begin{bmatrix} e^{2t}\sin(3t) \\ 2e^{2t}\cos(3t) + e^{2t}\sin(3t) \end{bmatrix}$

6. (17 pts) Consider the second order linear ODE:

$$y'' + 3y' + 2y = 0.$$

[5 points] (a) Rewrite this as a homogeneous linear system of first order ODEs: $\hat{x}' = \mathbb{A}\hat{x}$. Describe your choice of $\hat{x}$ and $\mathbb{A}$ explicitly.

$x_1 = y$ $\quad x_1' = y' = x_2$

$x_2 = y'$ $\quad x_2' = y'' = -3y' - 2y$

$= -3x_2 - 2x_1$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Answer: $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$

Answer: $\mathbb{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

[9 points] (b) Find the eigenvalues and corresponding eigenvectors of your matrix $\mathbb{A}$ from part (a).

$x^2 + 3x + 2 = (x+1)(x+2)$ $\quad -1, -2$

$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$

$\rightarrow \alpha = -1$ $\quad$ so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is eigenvector for -1

$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ \beta \end{bmatrix} = -2 \begin{bmatrix} 1 \\ \beta \end{bmatrix}$ $\quad$ so $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is eigenvector for -2

$\rightarrow \beta = -2$

Eigenvalues: $-1, -2$

Eigenvectors: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

[3 points] (c) Write down the general solution to the system, i.e., the general solution for $\hat{x}$.

Answer: $\hat{x} =$ $C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$