MTH 165: Linear Algebra with Differential Equations

Final Exam

December 17, 2016

NAME (please print legibly): Solutions

Your University ID Number:

Indicate your instructor with a check in the box:

Bobkova	MWF 10:25-11:15	
Lubkin	MWF 9:00-9:50	
Rice	TR 14:00-15:15	
Vidaurre	MW 14:00-15:15	

- You have 180 minutes to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed during this exam.
- **Show all your work and justify your answers, even if the answer is just a number or "yes/no" or "true/false".** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 13 pages.

QUESTION	VALUE	SCORE
1	20	
2	36	
3	24	
4	36	
5	36	
6	24	
7	24	
TOTAL	200	

1. (20 points) Solve the following initial value problems in explicit form.

10 (a) $y' = 4e^{2x-y}$, $y(0) = 0$

$e^y \cdot y' = 4e^{2x}$ 3 sep. var.

$\int e^y \, dy = \int 4e^{2x} \, dx$

$e^y = 2e^{2x} + C$ 3 integ.

$1 = 2 \cdot 1 + C$

$\to C = -1$ 2

$e^y = 2e^{2x} - 1$

$y = \ln(2e^{2x} - 1)$

2

10 (b) $y' = 4e^{2x} - y$, $y(0) = 0$

$y' + y = 4e^{2x}$ 2 std. form

$p(x) = 1$

$I(x) = e^{\int 1 dx} = e^x$ 2 int. factor.

$y = e^{-x} \int 4e^{3x}$

$= e^{-x}\left(\frac{4}{3}e^{3x} + C\right)$ 4 formula + integ.

$0 = 1\left(\frac{4}{3} + C\right) \to C = -\frac{4}{3}$ 2

$y = e^{-x}\left(\frac{4}{3}e^{3x} - \frac{4}{3}\right)$

Alt. Solution

$y' + y = 4e^{2x}$

$y_p = Ae^{2x}$

$y' + y = 3Ae^{2x} = 4e^{2x}$

$A = \frac{4}{3}$

$r + 1 = 0$

$\to r = -1$

$y = \frac{4}{3}e^{2x} + Ce^{-x}$

$0 = \frac{4}{3} + C \to C = -\frac{4}{3}$

$y = \frac{4}{3}e^{2x} - \frac{4}{3}e^{-x}$

2. (36 points) Determine if the following statements are true or false.

9 (a) The polynomials $p_1(x) = 1$, $p_2(x) = x^2 + x$, $p_3(x) = 2 + 5x$, and $p_4(x) = x^2 - 1$ are linearly independent vectors in $P_2(\mathbb{R})$, the space of polynomials of degree at most 2.

False, because the number of vectors (4) is greater than the dimension of the vector space (3).

Alternatively, they could find a specific dependence

- Some partial credit (3 pts?) for any attempt at a solution that displays knowledge of the defn. of linear independence.

9 (b) The vectors $\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

This is equivalent to the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 7 & -1 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$ having rank 3.

$$\det(A) = 1(2-0) + 3(28+2) = 92 \neq 0,$$

so true

- many alt. solutions available, though I expect most students will use determinant
- Partial credit for exhibiting knowledge of lin ind, spanning (maybe 2 pts each)

9 (c) The set $S = \{(x, y) : x + y \geq 0\}$ is a subspace of $\mathbb{R}^2$.

$(1,0) \in S$, but $(-1,0) = -(1,0) \notin S$,

so S is NOT closed under scaling,

so <u>false</u>

partial credit for showing $\vec{0} \in S$
and S closed under addition if
wrong conclusion is reached (maybe 2pts for each)

9 (d) The transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ c+d \\ ab \\ cd \end{bmatrix}$ is linear.

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} \neq 2\,T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right),$$ so <u>false</u>

- 2 pts for noting $\vec{0} \in S$ if wrong conclusion is reached
- wide variety of counterexamples possible, for addition or scaling

3. (24 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 3 & -5 \\ 0 & 2 & -3 \end{bmatrix}.$$

12 (a) Determine all eigenvalues of A.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 3-\lambda & -5 \\ 0 & 2 & -3-\lambda \end{vmatrix}$$

4 defn of char. poly.

$$= (1-\lambda)\left((3-\lambda)(-3-\lambda)+10\right)$$ 4 det. calc.

$$= (1-\lambda)(\lambda^2+1) = 0$$

$$\lambda = 1, \pm i$$ 4 roots

12 (b) If A^{-1} exists, compute it. If A^{-1} does not exist, explain why not.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ 0 & 2 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right] \xrightarrow[-3R_3 + R_1 \to R_1]{2R_3 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 6 & -9 \\ 0 & 1 & 0 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right]$$

6pts: Correct Gauss-Jordan procedure

6pts: execution (maybe 2 pts off per small error)

$$A^{-1} = \begin{bmatrix} 1 & 6 & -9 \\ 0 & -3 & 5 \\ 0 & -2 & 3 \end{bmatrix}$$

"Cramer's Rule" ok, too, small penalties for arithmetic errors

4. (36 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 3 & 6 & 9 & 16 & 22 \\ 5 & 10 & 15 & 26 & 36 \end{bmatrix},$$

and suppose the function $T : \mathbb{R}^5 \to \mathbb{R}^3$ is defined by $T(\mathbf{x}) = A\mathbf{x}$.

9 (a) Determine a row-echelon form of A.

$-3R_1 + R_2 \to R_2$

$-5R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[-R_2+R_3 \to R_3]{}$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5pts: procedure

4pts: execution

ONLY PENALIZE FOR ROW REDUCTION ERRORS ONCE!

if parts (b), (c), (d) correctly use answer to part (a), they get full credit.

9 (b) Determine a basis for the range (also known as the image) of T.

Range of T = column space of A

basis for column space = 1^{st} and 4^{th} columns of A

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 16 \\ 26 \end{bmatrix}$$

9 (c) What is the dimension of the kernel of T?

Rank-Nullity Theorem: $\dim(\mathrm{Rng}(T)) + \dim(\ker(T)) = \dim(\mathbb{R}^5)$

$2 + \dim(\ker(T)) = 5$

$\dim(\ker(T)) = 3$

Alt: Work with the matrix directly, conclude it has a 3-dim nullspace

9 (d) Are the rows of A linearly independent?

No, because the dimension of the span of the rows (2) is less than the number of rows (3).

Alt: Conclude that row operations display a dependence, or any other exhibition of a dependence is fine...

Remember: If they made an error in part a and found the matrix had rk 3, then concluded "yes", they get credit

5. (36 points)

9

(a) Find the general real-valued solution to the differential equation

$$y^{(4)} - 25y'' = 0.$$

aux. eqn.: $r^4 - 25r^2 = 0$ 2

$r^2(r^2 - 25) = 0$

$r = 0$ (mult. 2), ± 5 2

$$y = C_1 + C_2 x + C_3 e^{5x} + C_4 e^{-5x}$$

1 2 1 1

9

(b) Find a particular real-valued solution to the differential equation

$$y'' + 9y = 30e^x.$$

Guess $y_p = Ce^x$

$y_p'' + 9y_p' = 10Ce^x = 30e^x$

$C = 3$

$$y_p = 3e^x$$

Any way to arrive at this is fine.

9

(c) Find a particular real-valued solution to the differential equation

$$y'' + 9y = 12\sin(3x).$$

Guess: $y_p = Ax\sin(3x) + Bx\cos(3x)$ [$\sin(3x)$, $\cos(3x)$ are in the homog. solution]

3 form

Alt. Solution w/ "anninilator"?

$y_p' = A\sin(3x) + 3Ax\cos(3x) + B\cos(3x) - 3Bx\sin(3x)$

$y_p'' = 3A\cos(3x) + 3A\cos(3x) - 9Ax\sin(3x) + -3B\sin(3x) - 3B\sin(3x) - 9Bx\cos(3x)$ 3 derivatives

$\rightarrow y_p'' + 9y_p = 6A\cos(3x) - 6B\sin(3x) = 12\sin(3x)$

$\rightarrow A = 0, \quad B = -2$

$y_p = -2x\cos(3x)$ 3 final ans.

9

(d) Solve the initial value problem

$$y'' + 9y = 30e^x + 12\sin(3x), \quad y(0) = 4, \; y'(0) = 3.$$

homog. solution

aux. eqn.: $r^2 + 9 = 0$

$r = \pm 3i$

$y_1 = \cos(3x)$

$y_2 = \sin(3x)$

3 homog. basis functions

$y = 3e^x - 2x\cos(3x) + c_1\cos(3x) + c_2\sin(3x)$

3 gen. soln

$y(0) = 3 + c_1 = 4 \quad c_1 = 1$

$y' = 3e^x - 2\cos(3x) + 6x\sin(3x) + 3c_1\sin(3x) + 3c_2\cos(3x)$

$y'(0) = 3 - 2 + 3c_2 = 3$

$3c_2 = 2$

$c_2 = \frac{2}{3}$

3 solve for constants

$$y = 3e^x - 2x\cos x + \cos(3x) + \frac{2}{3}\sin(3x)$$

6. (24 points) Consider a spring-mass system with spring constant $k = 4$ N/m, and a mass $m = 1$ kg, which is hung from the spring and allowed to reach equilibrium. Determine the following about the position function y of the mass, relative to equilibrium, where the positive direction is downward.

12

(a) If the medium has damping constant $c = 4$, no external force is acting on the system, and the mass is stretched 1 m past equilibrium and released, find a formula for y.

$$my'' + cy' + ky = F(t)$$

$m=1$
$k=4$
$c=4$
$F(t)=0$
$y(0)=1$
$y'(0)=0$

4 $y'' + 4y' + 4y = 0$

aux eqn: $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r = -2$, mult. 2

4

$y = c_1 e^{-2t} + c_2 t e^{-2t}$

$y(0) = c_1 = 1$ $\quad y'(0) = -2c_1 + c_2 = 0$

$c_2 = 2$

$y = e^{-2t} + 2te^{-2t}$

4

12

(b) If there is no damping force, and an external force of $F(t) = \sin(t)$ acts on the system, describe all possible formulas (i.e. give the *general real-valued solution*) for y.

$m=1$
$k=4$
$c=0$
$F(t)=\sin(t)$

4 $y'' + 4y = \sin(t)$

<u>gen soln to $y''+4y=0$</u>

$r^2 + 4 = 0$

$r = \pm 2i$ 2

$y_1 = \cos(2t)$

$y_2 = \sin(2t)$

<u>part soln to $y''+4y = \sin(t)$</u>

Guess $y_p = A\sin(t) + B\cos(t)$

$y_p' = A\cos(t) - B\sin(t)$

$y_p'' = -A\sin(t) - B\cos(t)$

$y_p'' + 4y_p = 3A\sin(t) + 3B\cos(t) = \sin(t)$

$A = \frac{1}{3}$, $B = 0$

$y_p = \frac{1}{3}\sin(t)$ 2

<u>Ans</u>

$y = \frac{1}{3}\sin(t) + c_1\cos(2t) + c_2\sin(2t)$

4

7. (24 points)

12

(a) Solve the following initial value problem.

$$x' = 2x + y$$

$$y' = 2x + 3y$$

$$x(0) = 1,\ y(0) = 2.$$

$X' = AX$, $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ 2

eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4$

$= (\lambda - 4)(\lambda - 1) = 0$

$\lambda = 1, 4$ 2

eigenvectors:

$\lambda = 1$, $A - I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $v_1 + v_2 = 0$

choose $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ 2

$\lambda = 4$, $A - 4I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$ $-2v_1 + v_2 = 0$

choose $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 2

gen soln: $X = c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$X(0) = \begin{bmatrix} c_1 + c_2 \\ -c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow 3c_2 = 3$

$c_2 = 1$

$c_1 = 0$ 2

2

$x(t) = e^{4t}$

$y(t) = 2e^{4t}$

Vector-form final ans. is OK.

(b) Find the general, real-valued solution to the following system of equations.

$$x' = 3x + 2y$$

$$y' = -x + y.$$

$$X' = AX, \quad A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \quad 2$$

eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 2$

$$= \lambda^2 - 4\lambda + 5 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad 2$$

eigenvector: $\lambda = 2+i$

$$A - (2+i)I = \begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \to \begin{bmatrix} 1-i & 2 \\ 0 & 0 \end{bmatrix}$$

$(1-i)v_1 + 2v_2 = 0$

choose $\vec{v} = \begin{bmatrix} -2 \\ 1-i \end{bmatrix}$

3

$$e^{(2+i)t} \begin{bmatrix} -2 \\ 1-i \end{bmatrix} = e^{2t}\left(\cos(t) + i\sin(t)\right) \begin{bmatrix} -2 \\ 1-i \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} -2\cos(t) - 2i\sin(t) \\ \cos(t) - i\cos(t) + i\sin(t) + \sin(t) \end{bmatrix}$$

$$= \boxed{e^{2t} \begin{bmatrix} -2\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix}} + \boxed{e^{2t} \begin{bmatrix} -2\sin(t) \\ \sin(t) - \cos(t) \end{bmatrix}}\, i$$

3

gen soln: $X = c_1 X_1 + c_2 X_2$

$$\boxed{\begin{aligned} x &= -2c_1 e^{2t}\cos(t) - 2c_2 e^{2t}\sin(t) \\ y &= c_1 e^{2t}(\cos(t) + \sin(t)) + c_2 e^{2t}(\sin(t) - \cos(t)) \end{aligned}}$$

vector form final ans is OK

2