MATH 165

Final Exam

December 16, 2018

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Honor Pledge: "I affirm that I did not provide or receive any unapproved assistance during this exam." Sign here:

Circle your Instructor's Name along with the Lecture Time:

Jonathan Pakianathan (TR 2)   Rufei Ren (MW 2)
Kazuo Yamazaki (MW 12:30)   Ustun Yildirim (MW 9)

- **No notes, books, calculators or other electronics are allowed on this exam.**
- **Please SHOW ALL your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**
- **Please put your simplified final answers in the spaces provided.**

Part A		
QUESTION	VALUE	SCORE
1	22	
2	15	
3	15	
4	18	
5	15	
6	15	
TOTAL	100	

Part B		
QUESTION	VALUE	SCORE
1	17	
2	16	
3	17	
4	17	
5	16	
6	17	
TOTAL	100	

Part A

1. (22 pts)

[11 points] (a) Find the solution to the differential equation

$$x \ln(x) y' - y^2 = 1$$

which satisfies the initial condition $y(e) = 1$.

ANSWER: ______________________________

[11 points] (b) Find the general solution of the differential equation $xy' - 3y = x^8$.

ANSWER: ____

2. (15 pts)

Use Gauss-Jordan reduction to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ -2 & -2 & 5 \end{bmatrix}$$

if it exists.

ANSWER: ____________________

3. (15 pts) Consider a system of linear equations expressed as $\mathbb{A}\mathbf{x} = \mathbf{b}$ where $\mathbb{A}$ is a $m \times n$ matrix. Let $\mathbb{A}^{\#}$ denote the augmented matrix for the system, $r = rank(\mathbb{A})$ and $r^{\#} = rank(\mathbb{A}^{\#})$. In each of the following cases, what can be said about the number of solutions? **(Circle only one of the choices in each part.)**

1. If $r = r^{\#}$, then the system
   (a) is inconsistent.
   (b) has a unique solution.
   (c) has infinitely many solutions.
   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

2. If $r < r^{\#}$, then the system
   (a) is inconsistent.
   (b) has a unique solution.
   (c) has infinitely many solutions.
   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

3. If $m = n$ and $r = m$, then the system
   (a) is inconsistent.
   (b) has a unique solution.
   (c) has infinitely many solutions.
   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

4. If $m = n, \mathbf{b} = \mathbf{0}$, and $r < m$, then the system
   (a) is inconsistent.
   (b) has a unique solution.
   (c) has infinitely many solutions.
   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

5. If $m < n$ and $\mathbf{b} \neq \mathbf{0}$, then the system
   (a) is inconsistent.
   (b) has a unique solution.
   (c) has infinitely many solutions.
   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

4. (18 pts)

[8 points] (a) Find the determinant of

$$M = \begin{pmatrix} 3 & 2 & 0 \\ 5 & 2 & 1 \\ -1 & 7 & 0 \end{pmatrix}$$

ANSWER: ____

[10 points] (b) Suppose A is a 4×4 matrix with $\det(A) = 2$ and B is obtained from A by adding 5 times row 2 to row 3. Then:

(i) $\det(3A) =$ ____

(ii) $\det(A^T) =$ ____

(iii) $\det(A^{-1}) =$ ____

(iv) $\det(A^3) =$ ____

(v) $\det(B) =$ ____

5. (15 pts) Determine which of the following subsets of $\mathbb{P}_3$ are **subspaces** of $\mathbb{P}_3$. ($\mathbb{P}_3$ is the vector space of real polynomials of degree 3 or less.) For each subset, **circle** NO if it is not a subspace and list a subspace property that fails for this subset in the provided slot. **Circle** YES if it is a subspace and in this case find and enter the dimension of this subspace in the slot provided.

(a) $S_1 = \{p(t) \in P_3 \mid p'(t) + 2p(t) + 7 = 0 \text{ for all } t\}$

NO it is not a subspace. A subspace property that fails to hold is ____________________
YES it is a subspace and its dimension is _____

(b) $S_2 = \{p(t) \in P_3 \mid p(-t) = p(t) \text{ for all } t\}$

NO it is not a subspace. A subspace property that fails to hold is ____________________
YES it is a subspace and its dimension is _____

(c) $S_3 = \{p(t) \in P_3 \mid p(0) = 1\}$

NO it is not a subspace. A subspace property that fails to hold is ____________________
YES it is a subspace and its dimension is _____

(d) $S_4 = \{p(t) \in P_3 \mid p'''(t) = 0 \text{ for all } t\}$

NO it is not a subspace. A subspace property that fails to hold is ____________________
YES it is a subspace and its dimension is _____

(e) $S_5 = \{p(t) \in P_3 \mid p'(3) = p(1)\}$

NO it is not a subspace. A subspace property that fails to hold is ____________________
YES it is a subspace and its dimension is _____

6. (15 pts)

The reduced row echelon form of

$$A = \begin{pmatrix} 3 & -6 & 1 & 3 & 0 \\ 2 & -4 & 1 & -1 & 0 \\ 3 & -6 & 0 & 12 & 1 \end{pmatrix}$$

is

$$U = \begin{pmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

[3 points] (a) The rank of A is

ANSWER: ______________________________

[3 points] (b) The nullity of A is

ANSWER: ______________________________

[3 points] (c) List a set of basis vectors for the column space of A.

ANSWER: ______________________________

[3 points] (d) List a set of basis vectors for the null space of A.

ANSWER: ______________________________

[3 points] (e) Give an example of a nontrivial linear dependency amongst the columns of A.

ANSWER: ______________________________

Part B

1. (17 pts) For the differential equation

$$(D^2 + 1)^2(D + 2)y = x,$$

[7 points] (a) Find the general solution y_c to its associated homogeneous differential equation.

ANSWER: ______________________________

[7 points](b) Find a particular solution y_p to the differential equation.

ANSWER: ______________________________

[3 points](c) Determine the general solution to the differential equation.

ANSWER: ______________________________

2. (16 pts) Consider the 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

[4 points] (a) Determine the eigenvalues of A.

ANSWER: ______________________________

[8 points] (b) Determine the eigenspaces corresponding to each of the eigenvalues of A.

ANSWER: ______________________________

[4 points] (c) Determine if A is defective. Justify your answer.

ANSWER: ______________________________

3. (17 pts) Solve the initial value problem

$$y'' + 2y' + 5y = 0$$

with $y(0) = 1, y'(0) = 2$.

ANSWER: ______________________________

4. (17 pts) The motion of a certain physical system is described by

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -cx_1 - bx_2 \end{aligned}$$

where $b > 0, c > 0$ and $b > 2\sqrt{c}$ and the independent variable is time t.
[13 points] (a) Find the general solution for x_1 and x_2.

ANSWER: ____

[4 points] (b) What happens to the general solution in (a) as $t \to \infty$? Does it blow up or approach a certain limit? Justify your answer carefully.

ANSWER: ____

5. (16 pts)

[8 points] (a) Suppose a system $\hat{x}' = \mathbb{A}\hat{x}$ where $\mathbb{A}$ is a 2×2 matrix has general solution

$$\hat{x} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find $\mathbb{A}$.

ANSWER: ____

[8 points] (b) Let $\mathbb{B}$ be a 2×2 real matrix which has eigenvalue $2 + 3i$ with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 + 4i \end{bmatrix}$. Write down the general solution to $\hat{x}' = \mathbb{B}\hat{x}$ where the independent variable is time t. **Please make sure that the two basis solutions used in the final form of your general solution are real valued quantities**.

ANSWER: ____

6. (17 pts) Consider the second order linear ODE:

$$y'' + 5y' + 6y = 0.$$

[5 points] (a) Rewrite this as a homogeneous linear system of first order ODEs: $\hat{x}' = \mathbb{A}\hat{x}$. Describe your choice of $\hat{x}$ and $\mathbb{A}$ explicitly.

ANSWER: ____

[9 points] (b) Find the eigenvalues and corresponding eigenvectors of your matrix $\mathbb{A}$ from part (a).

ANSWER: ____

[3 points] (c) Write down the general solution to the system, i.e., the general solution for $\hat{x}$.

ANSWER: ____