MATH 165

Final Exam

December 16, 2018

NAME (please print legibly): Key

Your University ID Number: __________

Honor Pledge: "I affirm that I did not provide or receive any unapproved assistance during this exam." Sign here:

Circle your Instructor's Name along with the Lecture Time:

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- **No notes, books, calculators or other electronics are allowed on this exam.**
- **Please SHOW ALL your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**
- **Please put your simplified final answers in the spaces provided.**

Part A		
QUESTION	VALUE	SCORE
1	22	
2	15	
3	15	
4	18	
5	15	
6	15	
TOTAL	100	

Part B		
QUESTION	VALUE	SCORE
1	17	
2	16	
3	17	
4	17	
5	16	
6	17	
TOTAL	100	

Part A

1. (22 pts)

[11 points] (a) Find the solution to the differential equation

$$x \ln(x) y' - y^2 = 1$$

which satisfies the initial condition $y(e) = 1$.

$x \ln x \, y' = 1 + y^2$

$\frac{1}{1+y^2} y' = \frac{1}{x \ln x}$

$\int \frac{1}{1+y^2} dy = \int \frac{1}{x \ln x} dx$

let $u = \ln x$

$du = \frac{1}{x} dx$

$\int \frac{1}{u} du = \ln(u).$

$\arctan(y) = \ln(\ln x) + C$

$y = \tan(\ln(\ln x) + c)$

$y(e) = \tan(\ln(\ln e) + c)$

$= \tan(\ln(1) + c)$

$= \tan(c)$

$= 1$

$c = \frac{\pi}{4}$

$y = \tan\left(\ln(\ln x) + \frac{\pi}{4}\right).$

ANSWER: ____

[11 points] (b) Find the general solution of the differential equation $xy' - 3y = x^8$.

$y' - \frac{3}{x}y = x^7$

Let $I(x) = e^{-\int \frac{3}{x} dx} = e^{-3\ln x} = x^{-3}$

$x^{-3}y = \int x^{-3}x^7 dx$

$= \int x^4 dx$

$= \frac{x^5}{5} + c$

$y = \frac{x^8}{5} + x^3 c$

ANSWER: ______________________________

2. (15 pts)

Use Gauss-Jordan reduction to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ -2 & -2 & 5 \end{bmatrix}$$

if it exists.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ -2 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 7 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-2R_2]{R_1-2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow[R_2-3R_3]{R_1+5R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -12 & 5 \\ 0 & 1 & 0 & -6 & 7 & -3 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 11 & -12 & 5 \\ -6 & 7 & -3 \\ 2 & -2 & 1 \end{bmatrix}$$

ANSWER: ____________________

3. (15 pts) Consider a system of linear equations expressed as $\mathbb{A}\mathbf{x} = \mathbf{b}$ where $\mathbb{A}$ is a $m \times n$ matrix. Let $\mathbb{A}^\#$ denote the augmented matrix for the system, $r = rank(\mathbb{A})$ and $r^\# = rank(\mathbb{A}^\#)$. In each of the following cases, what can be said about the number of solutions? **(Circle only one of the choices in each part.)**

1. If $r = r^\#$, then the system

   (a) ~~is inconsistent.~~

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

2. If $r < r^\#$, then the system

   (a) is inconsistent.

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

3. If $m = n$ and $r = m$, then the system

   (a) is inconsistent.

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

4. If $m = n, \mathbf{b} = \mathbf{0}$, and $r < m$, then the system

   (a) is inconsistent.

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

5. If $m < n$ and $\mathbf{b} \neq \mathbf{0}$, then the system

   (a) is inconsistent.

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

4. (18 pts)

[8 points] (a) Find the determinant of

$$M = \begin{pmatrix} 3 & 2 & 0 \\ 5 & 2 & 1 \\ -1 & 7 & 0 \end{pmatrix}$$

$(-1)\begin{vmatrix} 3 & 2 \\ -1 & 7 \end{vmatrix} = (-1)(21+2) = -23$

ANSWER: ____________________

[10 points] (b) Suppose A is a 4×4 matrix with $\det(A) = 2$ and B is obtained from A by adding 5 times row 2 to row 3. Then:

(i) $\det(3A) =$ $3^4(2)$

(ii) $\det(A^T) =$ 2

(iii) $\det(A^{-1}) =$ $\frac{1}{2}$

(iv) $\det(A^3) =$ 8

(v) $\det(B) =$ 2

5. (15 pts) Determine which of the following subsets of $\mathbb{P}_3$ are **subspaces** of $\mathbb{P}_3$. ($\mathbb{P}_3$ is the vector space of real polynomials of degree 3 or less.) For each subset, **circle** NO if it is not a subspace and list a subspace property that fails for this subset in the provided slot. **Circle** YES if it is a subspace and in this case find and enter the dimension of this subspace in the slot provided.

(a) $S_1 = \{p(t) \in P_3 \mid p'(t) + 2p(t) + 7 = 0 \text{ for all } t\}$

NO it is not a subspace. A subspace property that fails to hold is zero vector not in S_1.
YES it is a subspace and its dimension is ____

(b) $S_2 = \{p(t) \in P_3 \mid p(-t) = p(t) \text{ for all } t\}$

$at^3 + \cancel{bt^2} + ct + \cancel{d} = -at^3 + \cancel{bt^2} - ct + \cancel{d}$

$\left.\begin{array}{l} 2a = 0 \\ 2c = 0 \end{array}\right\}$ $a = c = 0$. Then a basis is $\{t^2, 1\}$

NO it is not a subspace. A subspace property that fails to hold is ____
YES it is a subspace and its dimension is 2

(c) $S_3 = \{p(t) \in P_3 \mid p(0) = 1\}$

NO it is not a subspace. A subspace property that fails to hold is no zero vector
YES it is a subspace and its dimension is ____

(d) $S_4 = \{p(t) \in P_3 \mid p'''(t) = 0 \text{ for all } t\}$

$\left.\begin{array}{l} p(t) = at^3 + bt^2 + ct + d \\ p'(t) = 3at^2 + 2bt + c \\ p''(t) = 6at + 2b \\ p'''(t) = 6a = 0 \end{array}\right\}$ $S_4 = P_2$.

NO it is not a subspace. A subspace property that fails to hold is ____
YES it is a subspace and its dimension is 3

(e) $S_5 = \{p(t) \in P_3 \mid p'(3) = p(1)\}$

$3a(9) + 2b(3) + c = a + b + c + d$

$26a + 5b - d = 0$

$d = 26a + 5b$

Any S_5 has the form $at^3 + bt^2 + ct + 26a + 5b$

NO it is not a subspace. A subspace property that fails to hold is ____
YES it is a subspace and its dimension is 3

6. (15 pts)

The reduced row echelon form of

$$A = \begin{pmatrix} 3 & -6 & 1 & 3 & 0 \\ 2 & -4 & 1 & -1 & 0 \\ 3 & -6 & 0 & 12 & 1 \end{pmatrix}$$

is

$$U = \begin{pmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

[3 points] (a) The rank of A is

ANSWER: 3

[3 points] (b) The nullity of A is

ANSWER: 2

[3 points] (c) List a set of basis vectors for the column space of A.

ANSWER: $\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

[3 points] (d) List a set of basis vectors for the null space of A.

$x_1 = 2x_2 - 4x_4$

$x_3 = 9x_4$

$x_5 = 0$

$N = (2x_2 - 4x_4, x_2, 9x_4, x_4, 0)$

$= \text{span}\{(2,1,0,0,0), (-4,0,9,1,0)\}$

ANSWER:

[3 points] (e) Give an example of a nontrivial linear dependency amongst the columns of A.

ANSWER: $C_2 + 2C_1 = 0$

Part B

1. (17 pts) For the differential equation

$$(D^2 + 1)^2(D + 2)y = x,$$

[7 points] (a) Find the general solution y_c to its associated homogeneous differential equation.

$y_c = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x + C_5 e^{-2x}$

ANSWER: ______________________

[7 points](b) Find a particular solution y_p to the differential equation.

$y_p = A + Bx$

$(D^2+1)^2(D+2)(A+Bx)$

$= (D^2+1)^2(B + 2A + 2Bx)$

$= (D^2+1)(B+2A+2Bx) = B + 2A + 2Bx = x$

$2B = 1$

$B + 2A = 0$

$B = \frac{1}{2}$

$\frac{1}{2} = -2A$

$A = -\frac{1}{4}$

ANSWER: $y_p = -\frac{1}{4} + \frac{1}{2}x$

[3 points](c) Determine the general solution to the differential equation.

ANSWER: $y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x + C_5 e^{-2x} - \frac{1}{4} + \frac{1}{2}x$

2. (16 pts) Consider the 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

[4 points] (a) Determine the eigenvalues of A.

$p(\lambda) = (2-\lambda)(1-\lambda)^2$

ANSWER: $\lambda_1 = 1, \lambda_2 = 2$

[8 points] (b) Determine the eigenspaces corresponding to each of the eigenvalues of A.

$\lambda_1 = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_2 = 0$

$v_1 = (0, 0, 1)$.

$\lambda_2 = 2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$2x_1 = x_3$

$x_2 = 0$

$(1, 0, 2)$.

ANSWER: ____________________

[4 points] (c) Determine if A is defective. Justify your answer.

ANSWER: Defective. The dimension of E_{λ_1} is 1, but the multiplicity of λ_1 is 2.

3. (17 pts) Solve the initial value problem

$$y'' + 2y' + 5y = 0$$

with $y(0) = 1, y'(0) = 2.$

$$p(r) = r^2 + 2r + 5$$

$$r = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i$$

$$y = e^{-x}\left(c_1 \cos(2x) + c_2 \sin(2x)\right).$$

$$y(0) = c_1 = 1$$

$$y'(x) = -e^{-x}(\cos(2x) + c_2 \sin 2x)$$
$$+ e^{-x}(-2\sin(2x) + 2c_2 \cos 2x)$$

$$y'(0) = 1 + 2c_2 = 2$$

$$2c_2 = 1$$

$$c_2 = \frac{1}{2}$$

$$y = e^{-x}\left(\cos 2x + \frac{1}{2}\sin 2x\right).$$

ANSWER: ______________________________

4. (17 pts) The motion of a certain physical system is described by

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -cx_1 - bx_2 \end{aligned}$$

where $b > 0, c > 0$ and $b > 2\sqrt{c}$ and the independent variable is time t.
[13 points] (a) Find the general solution for x_1 and x_2.

$A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$ $\quad \begin{vmatrix} -\lambda & 1 \\ -c & -b-\lambda \end{vmatrix} = +\lambda(+b+\lambda)+c$

$= \lambda^2 + b\lambda + c$

$\lambda = \dfrac{-b \pm \sqrt{b^2-4c}}{2}$

$b > 2\sqrt{c}$ are positive, so
$\Rightarrow b^2 > 4c$
$\Rightarrow 2$ distinct real roots

$\lambda_1 = \dfrac{-b+\sqrt{b^2-4c}}{2}$

$A - \lambda_1 I = \begin{bmatrix} \frac{b-\sqrt{b^2-4c}}{2} & 1 \\ -c & -b-\left(\frac{-b+\sqrt{b^2-4c}}{2}\right) \end{bmatrix} \rightarrow \begin{bmatrix} \frac{b-\sqrt{b^2-4c}}{2} & 1 \\ 0 & 0 \end{bmatrix}$

$\dfrac{b-\sqrt{b^2-4c}}{2} x_1 = -x_2$ $\quad v_1 = (2, -b+\sqrt{b^2-4c})$

$\lambda_2 = \dfrac{-b-\sqrt{b^2-4c}}{2}$

$A - \lambda_2 I$ in reduced form:

$\begin{bmatrix} \frac{b+\sqrt{b^2-4c}}{2} & 1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1\left(\dfrac{b+\sqrt{b^2-4c}}{2}\right) = -x_2$

$v_2 = (2, -b-\sqrt{b^2-4c})$

ANSWER: $\vec{x}(t) = c_1 e^{\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)t} \begin{bmatrix} 2 \\ -b+\sqrt{b^2-4c} \end{bmatrix} + c_2 e^{\left(\frac{-b-\sqrt{b^2-4c}}{2}\right)t} \begin{bmatrix} 2 \\ -b-\sqrt{b^2-4c} \end{bmatrix}$

[4 points] (b) What happens to the general solution in (a) as $t \to \infty$? Does it blow up or approach a certain limit? Justify your answer carefully.

$b = \sqrt{b^2}$, because $b > 0$.
Then $b = \sqrt{b^2} > \sqrt{b^2-4c}$.
So $0 > -b + \sqrt{b^2-4c}$

$e^{(-b+\sqrt{b^2-4c})t} \rightarrow 0$
and
$e^{(-b-\sqrt{b^2-4c})t} \rightarrow 0$

So the solution $\rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ANSWER:

5. (16 pts)

[8 points] (a) Suppose a system $\hat{x}' = \mathbb{A}\hat{x}$ where $\mathbb{A}$ is a 2×2 matrix has general solution

$$\hat{x} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find $\mathbb{A}$.

Solution A

$\lambda_1 = 3, \ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2, \ v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a+2b \\ c+2d \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\left.\begin{aligned} a+b &= 3 \\ a+2b &= 2 \\ c+d &= 3 \\ c+2d &= 4 \end{aligned}\right\} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}.$$

– OR – (not covered in 524): Solution B

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A = QDQ^{-1} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

ANSWER: ____

[8 points] (b) Let $\mathbb{B}$ be a 2×2 real matrix which has eigenvalue $2 + 3i$ with corresponding eigenvector $\begin{bmatrix} 1 \\ 1+4i \end{bmatrix}$. Write down the general solution to $\hat{x}' = \mathbb{B}\hat{x}$ where the independent variable is time t. **Please make sure that the two basis solutions used in the final form of your general solution are real valued quantities.**

$$(\cos 3t + i \sin 3t) \begin{bmatrix} 1 \\ 1+4i \end{bmatrix} = \begin{bmatrix} \cos 3t + i \sin 3t \\ \cos 3t + i \sin 3t + 4i \cos 3t - 4 \sin 3t \end{bmatrix}$$

$$= \begin{bmatrix} \cos 3t \\ \cos 3t - 4 \sin 3t \end{bmatrix} + i \begin{bmatrix} \sin 3t \\ \sin 3t + 4 \cos 3t \end{bmatrix}$$

$$\vec{x} = e^{2t} \left(c_1 \begin{bmatrix} \cos 3t \\ \cos 3t - 4 \sin 3t \end{bmatrix} + c_2 \begin{bmatrix} \sin 3t \\ \sin 3t + 4 \cos 3t \end{bmatrix} \right)$$

ANSWER: ____

6. (17 pts) Consider the second order linear ODE:

$$y'' + 5y' + 6y = 0.$$

[5 points] (a) Rewrite this as a homogeneous linear system of first order ODEs: $\hat{x}' = \mathbb{A}\hat{x}$. Describe your choice of $\hat{x}$ and $\mathbb{A}$ explicitly.

$x_1 = y$

$x_2 = y'$

Then $x_1' = x_2$

$x_2' = -6x_1 - 5x_2$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \vec{x}$

ANSWER: ______________________________

[9 points] (b) Find the eigenvalues and corresponding eigenvectors of your matrix $\mathbb{A}$ from part (a).

This seems familiar.

We can use Q4

with $c = 6$ & $b = 5$

as long as $5 > 2\sqrt{6}$.

Since $6 = \frac{24}{4} < \frac{25}{4}$, this is true.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -cx_1 - bx_2 \end{aligned}$$

The solution we gave was $\vec{X}(t) = c_1 e^{\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)t} \begin{bmatrix} 2 \\ -b+\sqrt{b^2-4c} \end{bmatrix} + c_2 e^{\left(\frac{-b-\sqrt{b^2-4c}}{2}\right)t} \begin{bmatrix} 2 \\ -b-\sqrt{b^2-4c} \end{bmatrix}$

$$\frac{-b \pm \sqrt{b^2-4c}}{2} = \frac{-5 \pm \sqrt{25-24}}{2} = \frac{-5+1}{2}, \frac{-5-1}{2}$$

$= -2, -3$ eigenvalues.

We get $\vec{X}(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ -4 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

eigenvectors

ANSWER: ______________________________

[3 points] (c) Write down the general solution to the system, i.e., the general solution for $\hat{x}$.

ANSWER: ______________________________