Math 165 Written Homework 7

Due Friday. March 22 at 11:59 pm on gradescope

Problems

1. Prove that $W = \{p(x) \in P_1(\mathbb{R}) \mid 2p'(0) = p(1)\}$ is a subspace of $P_1(\mathbb{R})$. Determine a basis for W. Justify your answer. Determine $\dim(W)$.

2. Determine a basis for the following subspaces W of the given vector spaces V. (You do not need to prove that W is a subspace.)

   (a) $V = M_{3\times 3}(\mathbb{R})$. $W = \{A \in M_{3\times 3}(\mathbb{R}) \mid A^T = -A\}$.

   (b) $V = \mathbb{R}^4$, $W = \text{null}(A)$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 2 & -4 & -9 & 5 \end{bmatrix}$$

3. Determine whether the set $S = \{(1, 2, 0, -2), (1, 0, 3, 2), (0, -2, 1, 1)\} \subset \mathbb{R}^4$ is linearly independent. Then determine whether or not $\vec{b} = (1, 1, 1, 1)$ is in $\text{span}(S)$. Show your work to support your answer.

4. Let a set S consist of 5 vectors in $\mathbb{R}^5$. Suppose we create a matrix A such that the columns of A are the transposes of the vectors of S.

   (a) Suppose $\text{rank}(A) = 5$. Is S linearly independent? Does it span $\mathbb{R}^5$? Explain your answers, or explain why there is not enough information.

   (b) Suppose $\text{rank}(A) = 3$. Is S linearly independent? Does it span $\mathbb{R}^5$? Explain your answers, or explain why there is not enough information.

1. Prove that $W = \{p(x) \in P_1(\mathbb{R}) \mid 2p'(0) = p(1)\}$ is a subspace of $P_1(\mathbb{R})$. Determine a basis for W. Justify your answer. Determine $\dim(W)$.

Pf that W is a subspace.

(a) Let $p_0(x)$ be the zero vector in $P_1(\mathbb{R})$. That is, $p_0(x)$ is the constant function that sends all x to 0.

Then $2p_0'(x) = 0 = p(1)$, so $p_0(x) \in W$.

(b) Let $p_1(x), p_2(x) \in W$. Then $2(p_1+p_2)'(0) = 2p_1'(0) + 2p_2'(0)$
$= p_1(1) + p_2(1)$, because $p_1, p_2 \in W$.
$= (p_1+p_2)(1)$.

Hence W is closed under scalar multiplication.

(c) Let $p(x) \in W$ and $\lambda \in \mathbb{R}$.

Then $2(\lambda p)'(0) = (2\lambda)\, p'(0) = \lambda(2p'(0)) = \lambda(p(1)) = (\lambda p)(1)$.

So W is closed under scalar multiplication.

2. Determine a basis for the following subspaces W of the given vector spaces V. (You do not need to prove that W is a subspace.)

(a) $V = M_{3\times3}(\mathbb{R})$. $W = \{A \in M_{3\times3}(\mathbb{R}) \mid A^T = -A\}$.

(b) $V = \mathbb{R}^4$, $W = \text{null}(A)$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 2 & -4 & -9 & 5 \end{bmatrix}$$

(a) Since $A^T = -A$, it must be true that $a_{ii} = -a_{ii}$ for each i. So $a_{ii} = 0$. Further, $a_{ji} = -a_{ij}$. Then a general vector in W has this form:

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}.$$ Treating a_{12}, a_{13}, and a_{23} as free parameters,

we get the basis:

$$\beta = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

(b)

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 2 & -4 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 0 & -6 & -9 & 3 \end{bmatrix} \xrightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ Let $x_3 = s$, $x_4 = t$

Then $x_1 = \frac{3}{2}s - \frac{3}{2}t$

$x_2 = -\frac{3}{2}s + \frac{1}{2}t$

A general vector in W has this form:

$$\left(\frac{3}{2}s - \frac{3}{2}t,\ -\frac{3}{2}s + \frac{1}{2}t,\ s,\ t\right) = s\left(\frac{3}{2}, -\frac{3}{2}, 1, 0\right) + t\left(-\frac{3}{2}, \frac{1}{2}, 0, 1\right)$$

So $\beta = \left\{\left(\frac{3}{2}, -\frac{3}{2}, 1, 0\right), \left(-\frac{3}{2}, \frac{1}{2}, 0, 1\right)\right\}$.

OR $\{(3, -3, 2, 0), (-3, 1, 0, 2)\}$

3. Determine whether the set $S = \{(1, 2, 0, -2), (1, 0, 3, 2), (0, -2, 1, 1)\} \subset \mathbb{R}^4$ is linearly independent. Then determine whether or not $\vec{b} = (1, 1, 1, 1)$ is in span(S). Show your work to support your answer.

We can address both questions at the same time.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & -2 & 1 \\ 0 & 3 & 1 & 1 \\ -2 & 2 & 1 & 1 \end{array}\right] \xrightarrow[R_4+2R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & -2 & -1 \\ 0 & 3 & 1 & 1 \\ 0 & 4 & 1 & 3 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 4 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_4-4R_2]{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 5 & 3 \end{array}\right] \xrightarrow{\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 5 & 3 \end{array}\right] \xrightarrow{R_4-5R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{7}{4} \end{array}\right]$$

We see that A has rank 3, so S is independent.
We see that $(A|b)$ has rank 4, so the system is inconsistent. This means $\vec{b} \notin \text{Span}(S)$.

4. Let a set S consist of 5 vectors in $\mathbb{R}^5$. Suppose we create a matrix A such that the columns of A are the transposes of the vectors of S.

(a) Suppose rank(A) = 5. Is S linearly independent? Does it span $\mathbb{R}^5$? Explain your answers, or explain why there is not enough information.

(b) Suppose rank(A) = 3. Is S linearly independent? Does it span $\mathbb{R}^5$? Explain your answers, or explain why there is not enough information.

(a) Yes. The rref form of A will be I_5. So the columns of A are independent. Since it's a set of 5 independent vectors in a 5-dimensional space, S does span $\mathbb{R}^5$.

(b) No, as this means 2 columns are linear combinations of the others. An independent set of 3 vectors can not span a 5-dimensional space, so S does not span $\mathbb{R}^5$.