MATH 165: WRITTEN HW 9

DUE: FRIDAY, NOV 22, 11:59PM ON GRADESCOPE
UNIVERSITY OF ROCHESTER, FALL 2024

Problem 1. For each part below, determine if the given function T is linear or non-linear. If it is linear, prove it by verifying the necessary conditions. If it is not linear, provide an example showing that T fails to have the relevant property.

(a) $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by $T(y) = y'' - x^2 y' + e^x y$. Here $C^\infty(\mathbb{R})$ is the vector space of all infinitely differentiable functions on the real line.

(b) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(A) = A^2$.

(c) $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(A) = AB - BA$, where B is a fixed $n \times n$ matrix.

(d) $T : C^0(\mathbb{R}) \to \mathbb{R}$ defined by $T(f) = \int_0^5 f(x)e^{-x}\,dx$. Here $C^0(\mathbb{R})$ is the vector space of all continuous functions on the real line.

(e) $T : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \det(A)$.

Problem 2. (a) Let $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ be the matrix transformation corresponding to the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

This means $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^4$. Find a basis for the kernel of T_A and for the range of T_A.

(b) Let $L : P_3(\mathbb{R}) \to \mathbb{R}^2$ be the linear map $L(f) = (f(0), f(1))$. For example $L(x^2+1) = (1, 2)$. Find a basis for the kernel of L and for the range of L. (*Hint*: Find a basis for $\operatorname{Ker}(L)$ and the dimension of $\operatorname{Ker}(L)$ first. From the general Rank-Nullity Theorem, what is the dimension of $\operatorname{Rng}(L)$?)