MATH 165: WRITTEN HW 8

DUE: FRIDAY, NOV 8, 11:59PM ON GRADESCOPE
UNIVERSITY OF ROCHESTER, FALL 2024

Problem 1. (a) Let $A \in M_{m\times n}(\mathbb{R})$ be an $m \times n$ real matrix, and $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ be the column vectors of A (in order left to right). Also, let $\vec{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}^T \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$. Recall that

$$A\vec{c} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$$

(see Problem 2(a) in the previous Written HW). Use this to explain why $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ is linearly independent if and only if $A\vec{c} = \vec{0}$ has a unique solution $\vec{c} = \vec{0}$.

(b) Let

$$\vec{v_1} = (1, 1, 1, 3), \quad \vec{v_2} = (1, 0, 1, 1), \quad \vec{v_3} = (-1, 2, -1, 3)$$

be vectors in $\mathbb{R}^4$. Determine if $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent or dependent. If it is linearly dependent, find specific values of constants c_1, c_2, c_3, not all zero, such that $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$. (*Hint*: Construct and solve a 4×3 SLE.)

(c) Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

be vectors in $M_2(\mathbb{R})$. Determine if $\{A_1, A_2, A_3\}$ is linearly independent or dependent. If it is linearly dependent, find specific values of constants c_1, c_2, c_3, not all zero, such that $c_1A_1 + c_2A_2 + c_3A_3 = O$. (*Hint*: 'Identify' the matrices with suitable column vectors.)

(d) Determine if a subset

$$\{x + 1, x^2 + x, x^3 + 1, x^3 + x\}$$

of $P_3(\mathbb{R})$ is linearly independent or dependent. If it is linearly dependent, find specific values of constants c_1, c_2, c_3, c_4, not all zero, such that

$$c_1(x + 1) + c_2(x^2 + x) + c_3(x^3 + 1) + c_4(x^3 + x) = 0.$$

(*Hint*: 'Identify' the polynomials with suitable column vectors.)

Problem 2. (a) Explain why a collection of more than n vectors in $\mathbb{R}^n$ must be linearly dependent.

(b) Explain why a collection of less than n vectors in $\mathbb{R}^n$ must not be a spanning set for $\mathbb{R}^n$.

(c) Give an example of two vectors in $\mathbb{R}^3$ which are linearly independent and another example of two vectors in $\mathbb{R}^3$ which is linearly dependent. Thus in general, if you have less than n vectors in $\mathbb{R}^n$, they might or might not be linearly dependent.

(d) Give an example of four vectors in $\mathbb{R}^3$ which span $\mathbb{R}^3$ and another example of four vectors of $\mathbb{R}^3$ which do not span $\mathbb{R}^3$. Thus in general, if you have more than n vectors in $\mathbb{R}^n$, they might or might not span $\mathbb{R}^n$.

(e) Explain why a collection of n vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ in $\mathbb{R}^n$ is a basis if and only if the RREF of

$$A = \begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \\ | & | & & | \end{bmatrix}$$

is the identity matrix.