MATH 165: WRITTEN HW 7

DUE: FRIDAY, NOV 1, 11:59PM ON GRADESCOPE
UNIVERSITY OF ROCHESTER, FALL 2024

Problem 1. (a) Find a spanning set, consisting of 4 explicit matrices, for $M_2(\mathbb{R})$, the vector space of all 2×2 real matrices.

(b) Find a spanning set, consisting of 3 explicit matrices, for

$$W = \{A \in M_3(\mathbb{R}) : A^T = -A\},$$

the vector space of all 3×3 real skew-symmetric matrices.

(c) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Find a spanning set, consisting of two explicit 4-column vectors, for the null space of A. (*Hint*: Solve $A\vec{x} = \vec{0}$ using Gaussian elimination first.)

Problem 2. (a) Let $A \in M_{m\times n}(\mathbb{R})$ be an $m \times n$ real matrix, and $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ be the column vectors of A (in order left to right). Also, let $\vec{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}^T \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$. (Here we consider $\mathbb{R}^m$ and $\mathbb{R}^n$ as the vector space of column vectors.) Explain why

$$A\vec{c} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n},$$

and why $\vec{b}$ is in the linear span of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ if and only if $A\vec{c} = \vec{b}$ for some $\vec{c} \in \mathbb{R}^n$.

(b) Let

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 12 \\ 8 \\ -3 \end{bmatrix}$$

be vectors in $\mathbb{R}^4$. Use Part (a) to determine if $\vec{b}$ is in $\mathrm{Span}(\vec{v_1}, \vec{v_2}, \vec{v_3})$ and, if it is, represent $\vec{b}$ as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$ (that is, find $c_1, c_2, c_3 \in \mathbb{R}$ satisfying $\vec{b} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$) by constructing and solving a suitable SLE.

(c) Let

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 3 & 12 \\ 8 & -3 \end{bmatrix}$$

be vectors in $M_2(\mathbb{R})$. Determine if B is in $\mathrm{Span}(A_1, A_2, A_3)$ and, if it is, represent B as a linear combination of A_1, A_2, A_3. (*Hint*: 'Identify' the matrices with suitable column vectors.)