# MATH 165: WRITTEN HW 2

DUE: FRIDAY, SEP 20, 11:59PM ON GRADESCOPE
UNIVERSITY OF ROCHESTER, FALL 2024

**Problem 1.** For this entire problem, assume an object of mass $m$ **falls from rest**, starting at a point near the Earth's surface. We discussed the free-fall model with air resistance proportional to the velocity of the object (see section 1.4 of our text). For this problem we will investigate a free-fall model with air resistance proportional to the **square** of the velocity. We assume that the drag force is given as

$$F_r = \pm k v^2,$$

where $k > 0$ is a constant of proportionality depending on the size and shape of the object, as well as the density and viscosity of the air. The square model has been shown to be more appropriate for some objects at high speeds.

(a) Taking the positive direction downward (as in the linear air resistance model in section 1.4) and assuming $F_r$ is proportional to $v^2$ as mentioned above, model a differential equation for the velocity of the object $v(t)$, similar to equation 1.4.10 on page 38. *Warning: make sure your resistance term has the correct sign.*

(b) Find the general solution to the equation in part (a) and apply your initial condition to find the particular solution below, explicitly solving for $v(t)$ (show enough work to display how it is derived):

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}} t\right),$$

where

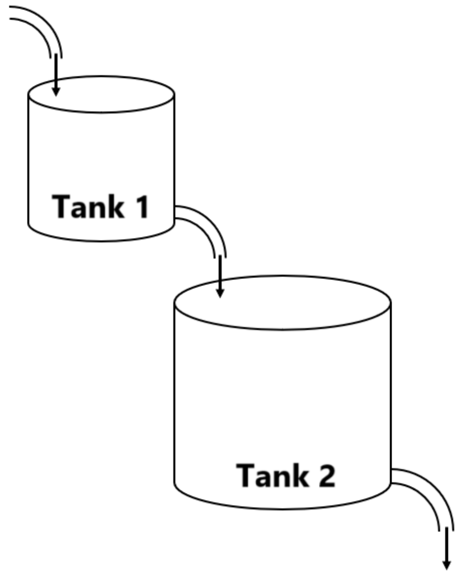
$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

is the hyperbolic tangent function. *Hint: The following integral formula should be helpful.*

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{u+a}{u-a}\right| + C.$$

(c) Take $\lim_{t\to\infty} v(t)$ to find the terminal velocity $v_T$. Show that $v = v_T$ is an equilibrium solution.

**Problem 2.** Consider the **cascade** of two tanks shown in the figure below.




Suppose that Tank 1 initially contains $V_1 = 100$ gal of brine (salt solution) and Tank 2 initially contains $V_2 = 200$ gal of brine. Each tank initially contains 50 lbs of salt. The three flow rates indicated in the figure are each 10 gal/min, with pure water flowing into Tank 1. Assume the tanks are well-mixed at all times, i.e., the density of salt in the brine solution is uniform throughout a given tank, at a given instant of time.

(a) Find the amount $x(t)$ of salt (measured in lbs) in Tank 1 at time $t$ (measured in minutes).

(b) Suppose that $y(t)$ is the amount of salt (measured in lbs) in Tank 2 at time $t$ (measured in minutes). Show first that
$$\frac{dy}{dt} = \frac{10x}{100} - \frac{10y}{200},$$
and then solve for $y(t)$, using the function $x(t)$ found in part (a).

(c) Find the maximum amount of salt ever in Tank 2.