MATH 165: WRITTEN HW 12

DUE: FRIDAY, DEC 13, 11:59PM ON GRADESCOPE
UNIVERSITY OF ROCHESTER, FALL 2024

Problem 1. Consider the simple harmonic oscillator system described in Section 8.5 of the book. This consists of a mass m attached to the end of a spring with spring constant k whose movement is across a medium with friction coefficient c. Here $m, k > 0$ and $c \geq 0$. The location of the mass, y (where $y = 0$ is the equilibrium position) as a function of time t satisfies the 2nd order linear DE:

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F_{\text{ext}}}{m}$$

where F_{ext} represents external forces during the motion other than friction and spring forces. Throughout this question we will ignore units and treat all these quantities as unitless quantities.

(a) Suppose $F_{\text{ext}} = 0$ and $c = 5, k = m = 2$. Is the system overdamped, critically damped or underdamped? Graph a representative solution (with some nonzero initial conditions) for this case - does the solution exhibit any oscillatory behaviour?

(b) Suppose $F_{\text{ext}} = 0$ and $c = 4, k = m = 2$. Is the system overdamped, critically damped or underdamped? Graph a representative solution (with some nonzero initial conditions) for this case - does the solution exhibit any oscillatory behaviour?

(c) Suppose $F_{\text{ext}} = 0$ and $c = k = m = 2$. Is the system overdamped, critically damped or underdamped? Graph a representative solution (with some nonzero initial conditions) for this case - does the solution exhibit any oscillatory behaviour?

(d) Suppose $F_{\text{ext}} = 0$ and $c = 0, k = m = 2$. Is the system overdamped, critically damped or underdamped? Graph a representative solution (with some nonzero initial conditions) for this case - does the solution exhibit any oscillatory behaviour?

(e) You attach a motor to the end of the spring which applies a periodic external driving force $F_{\text{ext}} = 10\cos(\omega t)$ to the system described in Part (d). For what value of frequency ω will this driving force cause the largest sized response (eventually at time t grows large) in the solutions of the system? What is the word used to describe this phenomenon?

Problem 2. Let x_1 be the population (in thousands) of spotted owls at time t in a certain ecosystem. Similarly let x_2 be the population (in thousands) of rats (one of the types of prey for spotted owls) in the same ecosystem as a function of t. Ecologists have observed after years of observing the habitat that these populations loosely satisfy a system of first order differential equations (where t is measured in decades):

$$\begin{aligned} x_1' &= \ \ -x_1 + x_2 \\ x_2' &= -2x_1 + x_2 + 4 \end{aligned}$$

(a) Writing $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the system of differential equations becomes $\vec{x}' = A\vec{x} + \vec{b}$ for some 2×2 matrix A and vector $\vec{b}$. Write down A and $\vec{b}$.

(b) Find the (complex) eigenvalues and eigenvectors of A.

(c) Find the general solution to the **homogeneous** system of differential equations $\vec{x}' = A\vec{x}$. It should be consisting only of real-valued functions.

(d) Find a particular solution $\vec{x_p}$ to the (nonhomogenous) system of differential equations of the form $\vec{x_p} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ where k_1, k_2 are constants (independent of time t).

(e) By the theory of systems of first order linear differential equations, it is known that the general solution to the system $\vec{x}' = A\vec{x} + \vec{b}$ is

$$\vec{x} = \vec{x_p} + \vec{x_c}$$

where $\vec{x_c}$ is the general solution to the associated homogeneous system found in Part (c). Use this to find the solution $\vec{x}(t)$ to the system $\vec{x}' = A\vec{x} + \vec{b}$ that satisfies $\vec{x}(0) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.