MATH 165: WRITTEN HW 11

DUE: FRIDAY, DEC 6, 11:59PM ON GRADESCOPE
UNIVERSITY OF ROCHESTER, FALL 2024

Problem 1. The goal of this exercise is to study Euler's important identity

$$e^{ix} = \cos(x) + \sin(x)i$$

and some of its consequences that will be useful in this course.

(a) Find the real and imaginary parts of the complex number $e^{(2+\frac{\pi}{4}i)}$.

(b) Note that $e^{ix}e^{iy} = e^{i(x+y)}$. Use this fact and Euler's identity to re-derive trigonometric formulas for $\sin(x+y)$ and $\cos(x+y)$ in terms of the quantities $\sin(x), \sin(y), \cos(x), \cos(y)$.

(c) For any $a, b \in \mathbb{R}$, show that $e^{(a+bi)x}$ and $e^{(a-bi)x}$ are in the linear span of

$$\{e^{ax}\cos(bx), e^{ax}\sin(bx)\}$$

by explicitly writing $e^{(a+bi)x}$ and $e^{(a-bi)x}$ as linear combinations (with complex coefficients) of $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$. Conversely, explain also how you can write $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ as linear combinations of $e^{(a+bi)x}$ and $e^{(a-bi)x}$, so $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ are in the linear span of

$$\{e^{(a+bi)x}, e^{(a-bi)x}\}.$$

This shows that

$$\text{Span}\{e^{(a+bi)x}, e^{(a-bi)x}\} = \text{Span}\{e^{ax}\cos(bx), e^{ax}\sin(bx)\}$$

If we consider them as complex vector spaces.

(d) Consider the quantity $B\cos(\mu t) + C\sin(\mu t)$ where B, C, μ are positive real numbers. Draw a right angle triangle with side lengths B, C and hypotenuse length $A = \sqrt{B^2 + C^2}$. Let θ be the angle (in radians) between the side of length B and the hypothenuse. Solve for B and C as functions of A and θ and use this to rewrite the quantity $B\cos(\mu t) + C\sin(\mu t)$ as a function of A, θ, μ, t, i.e., eliminate B and C. Then explain why

$$B\cos(\mu t) + C\sin(\mu t) = A\cos(\mu t - \theta).$$

A is called the **amplitude** of the final expression and θ is called the **phase**. This final form of the expression is often called the **Phase-Amplitude form**.

(e) Using Part (d), rewrite $3\cos(t) + 3\sin(t)$ in Phase-Amplitude form.

Problem 2. Consider the third-order constant coefficient homogeneous linear DE

$$y''' + 5y'' + 17y' + 13y = 0.$$

(a) Find the auxiliary polynomial $P(r)$ associated to this DE.

(b) Given that -1 is one of the roots to the auxiliary polynomial, we can write it as $P(r) = (r+1)Q(r)$ where $Q(r)$ is a quadratic polynomial which can be found by long division of polynomials. Carry out this computation to find all three roots of the auxiliary polynomial - two of the roots will be complex numbers which are complex conjugates of each other.

(c) The complex solutions to this DE form a 3-dimensional complex vector space spanned by the 3 (complex-valued) functions $\{e^{r_1 x}, e^{r_2 x}, e^{r_3 x}\}$ where r_1, r_2, r_3 are the three roots of the auxiliary polynomial. Use Euler's identity to find three **real-valued** functions $\{f_1, f_2, f_3\}$ which also span the same solution space.

(d) Find the general solution to this DE, i.e., the general form of the real-valued solutions to the DE.

(e) Find the general solution to the nonhomogeneous linear DE

$$y''' + 5y'' + 17y' + 13y = e^x.$$