MATH 165: WRITTEN HW 10

DUE: **MONDAY, DEC 2**, 11:59PM ON GRADESCOPE
UNIVERSITY OF ROCHESTER, FALL 2024

Problem 1. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(a) Compute the characteristic polynomial $p(x) = \det(A - xI)$.

(b) Compute the two eigenvalues of A. (You should get two complex eigenvalues conjugate to each other).

(c) Find an eigenvector for one of your eigenvalues obtained in Part (b).

(d) Note that when A is a real matrix, then if $A\vec{v} = (a+bi)\vec{v}$, by taking complex conjugates of both sides of this equation we find that $A\vec{w} = (a-bi)\vec{w}$ where $\vec{w}$ is the vector obtained from $\vec{v}$ by taking the complex conjugate of all its entries. Use this fact and your calculation in Part (c) to find the eigenvector for the other eigenvalue besides the one you computed in Part (c).

Problem 2. (a) Suppose that A is an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v_1}, \dots, \vec{v_n}$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Let S be the $n \times n$ matrix whose j-th column is v_j and Λ be the diagonal matrix whose diagonal entries are $\lambda_1, \dots, \lambda_n$ (in order, going from top left to bottom right). Explain briefly why S must be invertible and why $AS = S\Lambda$. (*Note.* It follows that $A = S\Lambda S^{-1}$, which is called the **diagonalization** of A. We do not cover diagonalization in this course, but for anyone interested in, see Section 7.3 in the textbook.)

(b) Let A be the 2×2 matrix with eigenvectors $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and corresponding eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 7$. Find A. (*Hint*: Observe that $\vec{v_1}$ and $\vec{v_2}$ are linearly independent, and form S and Λ as in Part (a).)

(c) Let B be a 3×3 matrix with eigenvalues $1, 2, 5$. Find the determinant of B. (*Hint*: What is Λ in Part (a) in this case?)

(d) Suppose $C\vec{v} = 2\vec{v}$ for some 4×4 matrix C and $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$. Find $(C^2 + 3C + 5I)(\vec{v})$.

(e) Compute all eigenvectors of $D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and show that there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of D, so D is a defective matrix. (*Note.* Do not use the notations of eigenspace and eigenbasis in this problem, even if you already learned them.)