MTH 165

Midterm 2

April 2, 2024

Name: Solutions

Student ID: ____________________

(If you do not know your SID, provide your netid.)

The Department of Mathematics adheres to the university's academic honesty policy. In addition, the following restrictions apply to this exam:

1. No phones, calculators or any other devices that could help you answer any part of this exam are permitted.

2. No notes or formula sheets or similar documents are permitted.

PLEASE COPY THE HONOR PLEDGE AND SIGN:
(Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

For your information: There are six problems on this exam.

1. (14 points) Let

$$A = \begin{bmatrix} 0 & 0 & -4 & 1 \\ -1 & 4 & -4 & 8 \\ 2 & 4 & -4 & -5 \\ 1 & 2 & 0 & -6 \end{bmatrix}.$$

B=

$$\begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 6 & -4 & 2 \\ 0 & 0 & -4 & 7 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

(a) Using elementary row operations, transform A to an upper triangular matrix B. Write your matrix B in the box provided.

$$A \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 0 & -6 \\ -1 & 4 & -4 & 8 \\ 2 & 4 & -4 & -5 \\ 0 & 0 & -4 & 1 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + R_1} \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 6 & -4 & 2 \\ 0 & 0 & -4 & 7 \\ 0 & 0 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 6 & -4 & 2 \\ 0 & 0 & -4 & 7 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

Grading note: There are infinitely-many correct solutions. If a student labeled row operations clearly, and if multiplying the determinant of B by the appropriate scalars resulting from clearly labeled EROs gave the correct determinant of A, full credit was awarded. If a student labeled row operations clearly, but the resulting determinant was not correct, 2/4 points was awarded.

This problem continues on the next page.

(b) Calculate the determinant of the matrix B you found in (a).

$(1)(6)(-4)(-6) = 144.$

Answer:
144

(c) Calculate the determinant of $\frac{1}{2}B$.

$\left(\frac{1}{2}\right)^4(144) = \frac{144}{16} = 9$

Answer: 9

(d) Calculate the determinant of the original matrix A.

Of the row operations performed in part (a), only the first — permuting rows 1 and 4 — affect determinant. That operation multiplies the determinant by -1.

Answer:
-144

2. (15 points)

Let $W = \{A \in M_{2\times 2}(\mathbb{R}) \mid \mathrm{trace}(A) = 0\}$. (Recall that trace$(A)$ is the sum of the diagonal entries of A.)

Show that W is a subspace of $M_{2\times 2}(\mathbb{R})$.

① $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has trace 0. So W is not empty.

② Suppose $A, B \in W$.

$tr(A+B) = tr(A) + tr(B) = 0 + 0 = 0.$

So W is closed under vector addition.

③ Suppose $A \in W$ and $\lambda \in \mathbb{R}$.

$tr(\lambda A) = \lambda\, tr(A) = \lambda(0) = 0.$

So W is closed under scalar multiplication.

Hence W is a subspace of $M_{2\times 2}(\mathbb{R})$.

Alternate proof:

If $A \in W$, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a+d=0$.

So $d=-a$. Then $W = \left\{ A \in M_2(\mathbb{R}) \mid A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \text{ for scalars } a,b,c \right\}$.

Hence $W = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

The span of a subset of V is always a subspace of V.

Really alternate proof.

$T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = \mathrm{trace}(A)$ is a linear transformation. Its kernel is W. The kernel of a linear transformation is a subspace of the domain.

3. (12 points) Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ t \\ 2 \end{bmatrix} \right\}$$

be a collection of vectors in $\mathbb{R}^4$.

(a) For what values of t is S a spanning set for $\mathbb{R}^4$?

Since $\dim(\mathbb{R}^4) = 4$, any spanning set must have at least 4 vectors.

Answer:
No possible t.

(b) For what values of t is S linearly independent?

Set $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 3 & 1 & t \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -2 & t-3 \\ 0 & 2 & 2 \end{bmatrix}$

$\xrightarrow[R_4 + 2R_2]{R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & t-1 \\ 0 & 0 & 0 \end{bmatrix}$. Then the columns are independent if and only if $t-1 \neq 0$.

Answer:
$t \neq 1$.

(c) In the case S is linearly **dependent**, find a basis for $Span(S)$.

If $t=1$, the last row operation above gives

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Hence columns 1 and 2 of A are independent.

Answer:
$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

4. (20 points) For each of the following subspaces W of the given vector spaces V, determine a basis β for W **and find its dimension**. You do not have to prove that W is a subspace. You should show enough work to convince a reader that you have provided an independent spanning set. Further proof is not required.

(a) $V = P_2(\mathbb{R})$ and $W = \{p(x) \in P_2(\mathbb{R}) \mid p''(0) = p'(2)\}$.

let $p(x) \in P_2(\mathbb{R})$. Set $p(x) = ax^2 + bx + c$.

$p'(x) = 2ax + b$.

$p'(2) = 4a + b$

$p''(x) = 2a$

$p''(0) = 2a$

If $2a = 4a + b$

then $b = -2a$

So a vector in W has the form

$ax^2 - 2ax + c$.

$\beta = \{x^2 - 2x, 1\}$.

This problem continues on the next page.

(b) (The instructions for this problem appear on the previous page.)

$V = M_{3\times 3}(\mathbb{R})$ and $W = \{A \in M_{3\times 3}(\mathbb{R}) \mid \text{trace}(A) = 0\}$

(Note: Trace is defined in Q2, but this V is the 3×3 matrices, not the 2×2 ones in Q2.)

A vector in W has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{11}-a_{22} \end{bmatrix}, \text{ as } \text{trace}(A) = a_{11} + a_{22} - a_{11} - a_{22} = 0.$$

Then we get

$$\beta = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

5. (15 points) Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ -1 & 2 & 3 & 0 & 1 \\ 0 & 2 & 2 & -2 & 1 \end{bmatrix}.$$

(a) Determine a basis for column space of A and **find its dimension**.

$$A \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & 5 & 1 & 1 \\ 0 & 2 & 2 & -2 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & 5 & 1 & 1 \\ 0 & 0 & -3 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow[-\frac{1}{3}R_3]{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[R_2 - \frac{5}{2}R_3]{R_1 - 2R_3} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

In this RREF form E, the first 3 columns are independent. Hence the first 3 columns of A are independent.

Alternative Solution. (Justification required for credit.)
Since rank(A) = 3, the column space of A is a 3-dimensional subspace of $\mathbb{R}^3$. So the columnspace is equal to $\mathbb{R}^3$. Then any basis for $\mathbb{R}^3$ will suffice.

Answer:

$$\beta = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\}$$

This problem continues on the next page.

(b) Determine a basis for null space of A and **find its dimension**.

$$E = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Set $x_4 = s, \quad x_5 = t$

$x_1 - s = 0$

$x_2 - 2s + \frac{1}{2}t = 0$

$x_3 + s = 0$

$\text{null}(A) = \{(s, 2s - \frac{1}{2}t, -s, s, t) \mid s, t \in \mathbb{R}\}$.

$\beta = \{(1, 2, -1, 1, 0), (0, -\frac{1}{2}, 0, 0, 1)\}$

Answer:

$\beta = \{(1, 2, -1, 1, 0), (0, -\frac{1}{2}, 0, 0, 1)\}$

dim $= 2$

(c) Show that your results in (a) and (b) satisfy the conclusion of the Rank-Nullity Theorem.

rank $A = 3$, $\dim(\text{null}(A)) = 2$, # of colums $= 5$.

$5 = 3 + 2$ ✓.

6. (24 points) Please mark each question true or false. Please fill in the box or use a check mark ***in*** the box rather than circling your answer. If you make a mistake, correct it and leave a note for your grader confirming your final answer.

Partial credit will not be offered for this problem. You do not need to justify your answer.

(a) The set of all 2×2 matrices A with real entries such that $\det A = 0$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

Not closed under addition.

☐ True

$\det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$ and $\det \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$, but

☒ False

$\det \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = 1.$

(b) The subset

$$V = \{A \in M_{2\times 2}(\mathbb{R}) | A + I \text{ is invertible}\}$$

is a subspace of $M_{2\times 2}(\mathbb{R})$ (Here I is the 2×2 identity matrix.)

Not closed under addition:

☐ True

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible

☒ False

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + I = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is invertible.

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not invertible.

(c) Let A be an $n \times n$ matrix such that $A^T A = I_n$, where I_n is the $n \times n$ identity matrix. Then $\det(A) = \pm 1$

Since $\det(A^T) = \det A$

☒ True

$\det(A^T A) = \det(A^2) = (\det A)^2.$

☐ False

If $(\det(A))^2 = \det(I_n) = 1,$

then $\det A = \pm 1.$

(d) Let $A \in M_{m\times n}(\mathbb{R})$ and $m < n$. Then the column space and the nullspace of A are subspaces of $\mathbb{R}^n$.

☐ True

☒ False

The columnspace is a subspace of $\mathbb{R}^m$.

(e) If an $m \times n$ matrix has rank n, then $m \geq n$.

☒ True

☐ False

rows $\geq$ rank

(f) A 3×4 matrix could have equal rank and nullity, but a 4×3 matrix can not.

☒ True

☐ False

Since rank + nullity = # columns, if rank = nullity, the # of columns must be even.

Scratch Work