MTH 165

Midterm 1

February 27, 2024

Name: Solutions

Student ID: ______________________

(If you do not know your SID, provide your netid.)

The Department of Mathematics adheres to the university's academic honesty policy. In addition, the following restrictions apply to this exam:

1. No phones, calculators or any other devices that could help you answer any part of this exam are permitted.
2. No notes or formula sheets or similar documents are permitted.

PLEASE COPY THE HONOR PLEDGE AND SIGN:
(Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________

For your information: There are seven problems on this exam.

1. (15 points) Solve the initial value problem. (Solve explicitly.) Circle your answer.

$$y\sin^2(t) + \sec(t)\frac{dy}{dt} = \sin^2(t)y^2, \qquad y(\pi) = 2$$

$$\sec t \frac{dy}{dt} = \sin^2 t\,(y^2 - y)$$

$$\int \frac{1}{y^2-y}\frac{dy}{dt} = \int \cos t \sin^2 t\, dt$$

$$\frac{1}{y^2-y} = \frac{1}{y-1} - \frac{1}{y}$$

$$\int \frac{1}{y-1} - \frac{1}{y}\, dy = \int \cos t \sin^2 t\, dt$$

let $u = \sin t$

$du = \cos t\, dt$

$\int u^2 du = \frac{u^3}{3} + C_1$

$$\ln|y-1| - \ln|y| = \frac{\sin^3 t}{3} + C$$

$$\ln\left|\frac{y-1}{y}\right| = \frac{\sin^3 t}{3} + C$$

$$\left|\frac{y-1}{y}\right| = e^{\frac{\sin^3 t}{3}} e^{C}$$

$$\frac{y-1}{y} = C_1 e^{\frac{\sin^3 t}{3}}$$

$$y - 1 = C_1 y e^{\frac{\sin^3 t}{3}}$$

$$y\left(1 - C_1 e^{\frac{\sin^3 t}{3}}\right) = 1$$

$$y = \frac{1}{1 - C_1 e^{\frac{\sin^3 t}{3}}}$$

$y(\pi) = 2$

$$2 = \frac{1}{C_1 e^0}$$

$2C_1 = 1$

$C_1 = \frac{1}{2}$

Ans:
$$y = \frac{1}{1 - \frac{1}{2}e^{\frac{\sin^3 t}{3}}}$$

2. (15 points)

(a) Find the general solution to the following first order differential equation:

$$y' + \frac{2y}{x} = x - 1.$$

(b) Find the solution curve that passes through the point $(1, 1)$?

Circle your answers.

$$I(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

$$x^2 y = \int x^2(x-1)\,dx$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + c$$

$$y = \frac{x^2}{4} - \frac{x}{3} + \frac{c}{x^2}$$

If $x=1$, $y=1$.

$$1 = \frac{1}{4} - \frac{1}{3} + c$$

$$c = \frac{13}{12}$$

$$y = \frac{x^2}{4} - \frac{x^3}{3} + \frac{13}{12x^2}$$

3. (20 points) Suppose a tank initially contains 40 litres of water in which is mixed 20 grams of salt. Suppose a solution flows into the tank at a rate of 1 litre per minute, and suppose the concentration of salt in this solution in 4 grams per liter. Suppose an additional source provides 1 litre of pure water to the tank each minute.Suppose a well-mixed solution flows out of the tank at a rate of 2 litres per minute.

(a) (10 pts) Determine a differential equation that gives the amount of salt as a function of time.

(b) (10 pts) Determine the concentration of salt in the tank as $t \to \infty$. Justify your answer.

(a) Concentration in is $\cancel{4 \text{ grams} / 2 \text{ liters}} = 2 g/L$

rate in $= 2 L/min$

rate out $= 2 L/min.$

$V(t) = V_0 + t(2-2) = 40$

—OR—

$\frac{dA}{dt} = \underbrace{(4 g/L)(1 L/m)}_{\text{solution}} + \underbrace{(0 g/L)(1 L/m)}_{\text{water}} - \frac{2}{40} A(t)$

$$\frac{dA}{dt} = (2)(2) - \frac{2}{40} A(t)$$

(b) $\frac{dA}{dt} + \frac{1}{20} A(t) = 4$

$I(t) = e^{\int \frac{1}{20} dt} = e^{\frac{t}{20}}$

$e^{\frac{t}{20}} A(t) = \int e^{\frac{t}{20}} 4\, dt$

$= (20)(4) e^{t/20} + C$

$A(t) = 80 + \frac{C}{e^{t/20}}$

$A(0) = 0 = 80 + C$

$C = -80$

$\lim_{t \to \infty} \frac{80\left(1 - e^{-t/20}\right)}{40} = 2 g/L.$

Answers:(a) $\frac{dA}{dt} = 4 - \frac{1}{20} A(t)$	(b) $2 g/L$

Page for scratchwork if needed

4. (10 points) Solve the system.

$$x_1 + x_2 = 1$$
$$x_2 + x_3 = 1$$
$$x_3 + x_4 = 1$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_2 - R_3]{R_1 + R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Let $x_4 = t$

$x_1 = 1 - t$

$x_2 = t$

$x_3 = 1 - t$

$S = \{(1,0,1,0) + t(-1,1,-1,1) \mid t \in \mathbb{R}\}$

Answer:

$S = \{(1,0,1,0) + t(-1,1,-1,1) \mid t \in \mathbb{R}\}$

Page for scratchwork if needed

5. (15 points)

Suppose a square matrix A is row-equivalent to a matrix B, where

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & -2 & 3 & -4 \end{bmatrix}.$$

(a) Determine the rank of A.

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & -2 & 3 & -4 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 3 & -4 \end{bmatrix} \xrightarrow{R_4+2R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_4-R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $B = 3$, so rank $A = 3$

(b) Is A invertible? Why or why not?

No. A is invertible if and only if its rank is 4.

(c) Does there exist a $\vec{b}$ such that $A\vec{x} = \vec{b}$ is must have infinitely-many solutions? Justify your answer.

Yes. Let $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. A homogeneous system always satisfies rank A = rank $A^{\#}$. If rank $A <$ number of variable, then the will be infinitely-many solutions.

(d) Does there exist a $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a unique solution? Justify your answer.

No. Since A is not invertible, $A\vec{x} = \vec{b}$ has no solutions ore infinitely-many.

A note about the term "unique." This means that for some particular $\vec{b}$ exactly one solution is possible. It does not refer to individual solutions. For example if $Ax = \vec{0}$, then $\vec{0} = (0,0,0,0)$ is a solution. But there are infinitely-many other solutions to that equation, so it is not unique.

6. (10 points) Write all the values of t for which the following matrix is invertible.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ t^2 & e^t & 3t^2 \\ e^{-t} & 1 & e^{t^2} \end{bmatrix}.$$

Determinant Method

$|A| = 1 \cdot \begin{vmatrix} t^2 & e^t \\ e^{-t} & 1 \end{vmatrix} = t^2 - 1.$ So if $t = 1$ or -1, $\det(A) = 0$.

Hence A is invertible if $t \neq \pm 1$.

Row Reduction method

$$\begin{bmatrix} 0 & 0 & 1 \\ t^2 & e^t & 3t^2 \\ e^{-t} & 1 & e^{t^2} \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} e^{-t} & 1 & e^{t^2} \\ t^2 & e^t & 3t^2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{e^t R_1} \begin{bmatrix} 1 & e^t & e^{t^2} \\ t^2 & e^t & 3t^2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 - t^2 R_1} \begin{bmatrix} 1 & e^t & e^{t^2} \\ 0 & e^t(1-t^2) & 3t^2 - e^{t^2+t} \\ 0 & 0 & 1 \end{bmatrix}$$

A is invertible if and only if $e^t(1-t^2) \neq 0$.

Since $e^t \neq 0$, A is invertible if $1 - t^2 \neq 0$, or $t \neq \pm 1$.

7. (15 points) Partial credit will not be offered for this problem. You do not need to justify your answer.

(a) Consider an $m \times n$ linear system withe coefficient matrix A and augmented matrix $A^{\#}$. If $Rank(A) = Rank(A^{\#})$, then the system has a unique solution.

☐ True

☒ False

There could be infinitely-many solutions.

(b) If A and B are invertible $n \times n$ matrices, then $(A+B)^2$ is invertible.

☐ True

☒ False

Consider I_n, $-I_n$. Both are invertible, but their sum is O_n.

(c) If A and B are 2×2 square matrices, then

$(A+B)^2 = A^2 + 2AB + B^2$

☐ True

☒ False

$(A+B)^2 = A^2 + AB + BA + B^2$

(d) If A is an $m \times n$ matrix, and if the rank of A is n, then the first n rows of a reduced row echelon form of A is the $n \times n$ identity matrix.

☒ True

☐ False

If rank $A = n$, $m \geq n$. So the matrix will be I_n with $m-n$ rows of zeros below.

(e) Suppose a linear system of m equations in n variables where $m > n$ has a coefficient matrix of rank n. Then the system has a unique solution.

☐ True

☒ False

The system could have no solutions.

Consider
$$\begin{aligned} x + y &= 1 \\ y &= 2 \\ x + y &= 3 \end{aligned}$$