MTH 165: Linear Algebra with Differential Equations

Final Exam

May 4, 2015

NAME (please print legibly): Solutions

Your University ID Number: ______________________

Indicate your instructor with a check in the box:

Dummit	TR 16:50-18:05	
Friedmann	MW 16:50-18:05	
Petridis	MWF 10:25-11:15	
Rice	MW 14:00-15:15	

- You have 3 hours to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed during this exam.
- Show all your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 14 pages.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. (10 points) Find a solution (implicit solutions are acceptable) for the following initial value problems on the domain $(0, \infty)$:

(a) $2y + xy' = x^{-1}$, $y(1) = A$.

$$y' + \frac{2}{x}y = x^{-2}$$

$$p(x) = \frac{2}{x}$$

$$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$y = \frac{1}{x^2}\int \frac{1}{x^2}\cdot\frac{1}{x^2}\,dx = \frac{1}{x^2}\int \frac{1}{x^4}\,dx = x^{-2}\left(\frac{-x^{-3}}{3} + c\right)$$

$$A = 1\left(-\frac{1}{3} + c\right) \Rightarrow c = A + \frac{1}{3}$$

$$y = x^{-2}\left(-\frac{x^{-3}}{3} + A + \frac{1}{3}\right)$$

(b) $2x + yy' = x^{-1}$, $y(1) = B$.

$$y \cdot y' = \frac{1}{x} - 2x$$

$$\int y\,dy = \int \frac{1}{x} - 2x\,dx$$

$$\frac{1}{2}y^2 = \ln(x) - x^2 + C$$

$$\frac{1}{2}B^2 = 0 - 1 + C$$

$$C = \frac{B^2}{2} + 1$$

$$\frac{1}{2}y^2 = \ln(x) - x^2 + \frac{B^2}{2} + 1$$

2. (10 points) Find a basis for the nullspace of each matrix.

(a) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$x_2 = 0$ Let $x_1 = r$, $x_3 = s$, $x_4 = t$

$\text{null}(A) = \left\{ \begin{bmatrix} r \\ 0 \\ s \\ t \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} t \right\}$

$\boxed{\text{Ans: } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$

(b) $B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

$x_1 - x_2 = 0$

$x_3 = 0$

Let $x_2 = s \rightarrow x_1 - s = 0,\ x_1 = s$

$x_4 = t$

$\text{null}(A) = \left\{ \begin{bmatrix} s \\ s \\ 0 \\ t \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} t \right\}$

Ans: $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

$x_1 + x_2 + x_3 + x_4 = 0$

$x_2 + x_3 + x_4 = 0$

$x_3 + x_4 = 0$

Let $x_4 = t$

$\rightarrow x_3 + t = 0$

$\rightarrow x_3 = -t$

$\rightarrow x_2 = x_1 = 0$

$\text{null}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ -t \\ t \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} t \right\}$

Ans: $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

3. (10 points) Let $M = \begin{bmatrix} k & 0 & k \\ 0 & 1 & 0 \\ 1 & 0 & k \end{bmatrix}$, where k is a parameter.

(a) Find $\det(M)$.

using column 2,

$\det(M) = 1(k^2 - k) = k^2 - k$

(b) Find all value(s) of k such that M is not an invertible matrix.

need $k^2 - k = k(k-1) = 0$,

so $k = 0, k = 1$

We continue taking $M = \begin{bmatrix} k & 0 & k \\ 0 & 1 & 0 \\ 1 & 0 & k \end{bmatrix}$, where k is a parameter.

(c) Find all value(s) of k such that $\lambda = 2$ is an eigenvalue of A.

$$\det(A-2I) = \begin{vmatrix} k-2 & 0 & k \\ 0 & -1 & 0 \\ 1 & 0 & k-2 \end{vmatrix}$$

$$= (-1)\left[(k-2)^2 - k\right]$$

$$= (-1)\left[k^2 - 4k + 4 - k\right]$$

$$= (-1)\left[k^2 - 5k + 4\right]$$

$$= (-1)(k-4)(k-1)$$

$k = 4, 1$

4. (10 points) Determine whether each given set S is a subspace of the given vector space V. If so, give a proof; if not, explain why not.

(a) $V = \mathbb{R}^3$ and $S = \{(x, y, z) \in V \mid x + y = z\}$.

① $(0,0,0) \in S$ since $0+0=0$.

② if $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S$,

then $(x_1+x_2)+(y_1+y_2) = (x_1+y_1)+(x_2+y_2) = z_1+z_2$,

so S is closed under addition.

③ if $(x,y,z) \in S$ and $\lambda \in \mathbb{R}$, then

$\lambda x + \lambda y = \lambda(x+y) = \lambda z$,

so S is closed under scaling.

Therefore, S IS a subspace.

(b) $V = M_2(\mathbb{R})$, the set of 2×2 matrices, and $S = \{A \in V \mid A^2 = 0\}$.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

are in S but $(A+B)^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

so S is NOT closed under addition,

hence S is NOT a subspace.

5. (10 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

(a) Find the eigenvalues of A, and determine (with justification) whether A is a defective matrix. (In other words, determine whether $\mathbb{R}^4$ has a basis consisting of eigenvectors of A.)

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 0 & 2-\lambda & 2 & 2 \\ 0 & 0 & 3-\lambda & 3 \\ 0 & 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda) = 0,$$

$$\rightarrow \quad \lambda = 1, 2, 3, 4$$

Since A has 4 distinct eigenvalues, they each have 1-dimensional eigenspaces and A is <u>NOT</u> defective.

(b) Find the eigenvalues of A^2, and determine (with justification) whether A^2 is a defective matrix. (In other words, determine whether $\mathbb{R}^4$ has a basis consisting of eigenvectors of A^2.)

Since $1, 2, 3, 4$ are eigenvalues of A,

we know that

$$1 \cdot 1 = 1$$
$$2 \cdot 2 = 4$$
$$3 \cdot 3 = 9$$
$$4 \cdot 4 = 16$$

are eigenvalues of $A \cdot A = A^2$, and hence A^2 is <u>not</u> defective by the identical reasoning as part (a).

6. (10 points) Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 real matrices. Consider the linear transformation $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d & b-c \\ a-c & b+d \end{bmatrix}.$$

(a) Find a basis for the kernel of T, and the dimension of the kernel.

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \quad \begin{aligned} a+d=0 &\rightarrow a=-d \\ b-c=0 &\rightarrow b=c \\ a-c=0 &\rightarrow a=c \\ b+d=0 &\rightarrow b=-d \end{aligned}$$

This means that a, b, c are all equal, and all equal to the opposite of d, i.e. a matrix of the form

$$\begin{bmatrix} t & t \\ t & -t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} t,$$

So the kernel has dimension 1 with basis $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Recall that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d & b-c \\ a-c & b+d \end{bmatrix}.$$

(b) Find the dimension of the range of T.

By the Rank-Nullity theorem,

$$\dim(\mathrm{Rng}(T)) + \dim(\ker(T)) = \dim(M_{2\times 2}(\mathbb{R})) = 4,$$

so by part (a)

$$\dim(\mathrm{Rng}(T)) + 1 = 4, \text{ i.e. } \boxed{\dim(\mathrm{Rng}(T)) = 3}$$

(c) Is the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in the range of T? Justify why or why not.

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \begin{aligned} a+d &= 1 \\ b-c &= 0 \\ a-c &= 0 \\ b+d &= 1 \end{aligned}$$

$$\Longrightarrow a = b = c = 1-d$$

Take $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and we see

$$T\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so YES

7. (10 points) Find the general solution for each differential equation:

(a) $y'' + 4y' + 4y = 0.$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r = -2$, mult. 2

$y = c_1 e^{-2t} + c_2 t e^{-2t}$

(b) $y^{(4)} - y = 0.$

$y = c_1 e^{t} + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$

$r^4 - 1 = 0$

$(r^2-1)(r^2+1) = 0$

$r = \pm 1, \pm i$

(c) $y''' - 2y'' + 5y' = 0.$

$y = c_1 + c_2 e^{t} \cos(2t) + c_3 e^{t} \sin(2t)$

$r^3 - 2r^2 + 5r = 0$

$r(r^2 - 2r + 5) = 0$

$r = 0, \quad \frac{2 \pm \sqrt{4-20}}{2}$

$r = 0, 1 \pm 2i$

8. (10 points) Solve the equation

$$y'' + 4y = 4\cos(2x) + 8e^{2x}$$

with initial conditions $y(0) = 3$, $y'(0) = 4$.

Gen. Soln to $y''+4y=0$

$r^2+4=0$ $\quad y_1 = \cos(2x)$

$r = \pm 2i$ $\quad y_2 = \sin(2x)$

Part. Soln to $y''+4y = 8e^{2x}$

guess $y = Ce^{2x}$, so

$y' = 2Ce^{2x}$

$y'' = 4Ce^{2x}$

$y''+4y = 4Ce^{2x} + 4Ce^{2x} = 8Ce^{2x}$

$= 8e^{2x}$

$\longrightarrow C = 1$

$y_{p_1} = e^{2x}$

$y = e^{2x} + x\sin(2x) + c_1\cos(2x) + c_2\sin(2x)$

$y(0) = 1 + c_1 = 3 \rightarrow c_1 = 2$

$y'(0) = 2 + 2c_2 = 4 \rightarrow c_2 = 1$

$y = e^{2x} + x\sin(2x) + 2\cos(2x) + \sin(2x)$

Part. Soln to $y''+4y = 4\cos(2x)$

Guessing $y = A\cos(2x) + B\sin(2x)$ won't work because that's the homogeneous solution, so instead we guess

$y = Ax\cos(2x) + Bx\sin(2x)$

$y_{p_2} = x\sin(2x)$

$y' = A\cos(2x) - 2Ax\sin(2x) + B\sin(2x) + 2Bx\cos(2x)$

$y'' = -2A\sin(2x) - 2A\sin(2x) - 4Ax\cos(2x) + 2B\cos(2x) + 2B\cos(2x) - 4B\sin(2x)$

$= -4A\sin(2x) + 4B\cos(2x) - 4Ax\cos(2x) - 4Bx\sin(2x)$,

so $y''+4y = -4A\sin(2x) + 4B\cos(2x) = 4\cos(2x)$

$\Longrightarrow A=0,\ B=1$

9. (10 points) Consider a spring-mass system with spring constant $k = 4\,\mathrm{N/m}$ and a mass $m = 1\,\mathrm{kg}$.

(a) Suppose there is no friction (or damping), and an external driving force of $6\sin(4t)\,\mathrm{N}$ is applied to the mass (in the positive direction). If at time $t = 0$ the mass is at rest in the equilibrium position, find the position $y(t)$ of the mass at time t for $t \geq 0$.

$$my'' + cy' + ky = F(t)$$

$$y'' + 4y = 6\sin(4t)$$

Part. Soln to nonhom.

Guess $y_p = A\sin(4t) + B\cos(4t)$

$y_p' = 4A\cos(4t) - 4B\sin(4t)$

$y_p'' = -16A\sin(4t) - 16B\cos(4t)$

$y_p'' + 4y_p = -12A\sin(4t) - 12B\cos(4t) = 6\sin(4t)$

$\to A = -\frac{1}{2},\ B = 0$

Gen Soln to hom.

$r^2 + 4 = 0$

$r^2 = -4,\ r = \pm 2i$

$y_1 = \cos(2t),\ y_2 = \sin(2t)$

$$y = -\frac{1}{2}\sin(4t) + \sin(2t)$$

$y = -\frac{1}{2}\sin(4t) + c_1\cos(2t) + c_2\sin(2t)$

$y(0) = c_1 = 0 \to c_1 = 0$

$y'(0) = -2 + 2c_2 = 0 \to c_2 = 1$

Continue to consider the spring-mass system with spring constant $k = 4\,\mathrm{N/m}$, a mass $m = 1\,\mathrm{kg}$, and an external driving force of $6\sin(4t)\,\mathrm{N}$ and no friction (or damping).

(b) What is the earliest time that the mass returns to its equilibrium position? (Hint: you may need to use the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.)

$$-\frac{1}{2}\sin(4t) + \sin(2t) = 0$$

$$-\sin(2t)\cos(2t) + \sin(2t) = 0$$

$$\underbrace{\sin(2t)}\left(1 - \cos(2t)\right) = 0$$

first 0 when $t = \frac{\pi}{2}$

first 0 when $t = \pi$

$$t = \frac{\pi}{2}$$

10. (10 points) Solve the system of differential equations

$$\begin{aligned} x_1' &= 2x_1 + 2x_2 \\ x_2' &= -x_1 + 4x_2 \end{aligned}$$

subject to the initial conditions $x_1(0) = 1$ and $x_2(0) = 1$.

$X' = AX$, where $A = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$

eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 10 = 0$

$$\lambda = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i$$

eigenvector: $\lambda = 3+i$

$A - (3+i)I = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \to \begin{bmatrix} -(1+i) & 2 \\ 0 & 0 \end{bmatrix}$

$-(1+i)v_1 + 2v_2 = 0$

choose $\vec{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$

$e^{(3+i)t} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = e^{3t}(\cos(t) + i\sin(t)) \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = e^{3t} \begin{bmatrix} 2\cos(t) + 2i\sin(t) \\ \cos(t) + i\cos(t) + i\sin(t) - \sin(t) \end{bmatrix}$

$$= \underbrace{e^{3t} \begin{bmatrix} 2\cos(t) \\ \cos(t) - \sin(t) \end{bmatrix}}_{X_1} + \underbrace{e^{3t} \begin{bmatrix} 2\sin(t) \\ \cos(t) + \sin(t) \end{bmatrix}}_{X_2} i$$

Gen. Soln: $X = c_1 X_1 + c_2 X_2$

$X(0) = \begin{bmatrix} 2c_1 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow c_1 = c_2 = 1/2$

$$\boxed{\begin{aligned} x_1(t) &= e^{3t}(\cos(t) + \sin(t)) \\ x_2(t) &= e^{3t}\cos(t) \end{aligned}}$$