MTH 165

Midterm 2

04/04/2020

Name: Key

UR ID: ______________________

Circle your Instructor's Name:

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- **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work or justification is given. Please record your final answers in the answer boxes.**
- **You are responsible for checking that this exam has all 6 problems.**

HONOR PLEDGE:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

YOUR SIGNATURE:______________________

1. (10 points) Let A, B and $C \in M_{n\times n}(\mathbb{R})$ be such that C is an invertible matrix and $AB = C^3$. Show that B is an invertible matrix.

2. (10 points) Given that

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 1,$$

find the following determinant:

$$\det \begin{bmatrix} 2a+d & 2b+e & 2c+f \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{bmatrix}.$$

Completely justify your answer.

3. (20 points) For the following vector spaces V, determine whether or not the given set W is a subspace of V. In either case, justify your answer thoroughly.

(a) $V = \mathbb{R}^4$ and $W = \{(x_1, x_2, x_3, x_4) \in V \,|\, x_1 + x_2 = x_3 + x_4\}$;

(b) $V = P_4(\mathbb{R})$ (i.e., the vector space of polynomials with real coefficients, having degree at most equal to 4) and $W = \{p \in V \,|\, p(1)p(2) = 2p(3)\}$.

4. (10 points) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Determine whether $A \in \operatorname{span}\{A_1, A_2, A_3\}$ or not. If so, express A as a linear combination of A_1, A_2, and A_3. Otherwise, explain why this is not possible.

5. (10 points) Let $V = M_{3\times 2}(\mathbb{R})$ and consider the subspace

$$W = \{A \in V \,|\, a_{11} + a_{22} = 3a_{12} + 2a_{32}\}.$$

Find a spanning set for W. Completely justify your answer.

6. (10 points) Is the set of polynomials

$$\{x^3 - x, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6\} \subseteq P_3(\mathbb{R})$$

linearly independent? If so, justify why; if not, find an explicit linear dependence.

1. (10 points) Let A, B and $C \in M_{n\times n}(\mathbb{R})$ be such that C is an invertible matrix and $AB = C^3$. Show that B is an invertible matrix.

Suppose B is not invertible. Then $\det B = 0$, so $\det AB = (\det A)(\det B) = 0$. Since C is invertible, $\det(C) = k \neq 0$. So $\det(C^3) = (\det(C))^3 = k^3$. Hence, if B is not invertible, $0 = k^3$. Then B must be invertible.

2. (10 points) Given that

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 1,$$

find the following determinant:

Set $B =$ this matrix.

$$\det \begin{bmatrix} 2a+d & 2b+e & 2c+f \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{bmatrix}.$$

Completely justify your answer.

$$= \det\begin{pmatrix} 2a & 2b & 2c \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{pmatrix} + \det\begin{pmatrix} d & e & f \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{pmatrix}$$

$$= (R_3 - \tfrac{1}{2}R_1)\begin{pmatrix} 2a & 2b & 2c \\ 2d+g & 2e+h & 2f+i \\ 2g & 2h & 2i \end{pmatrix} + (R_2 - 2R_1)\det\begin{pmatrix} d & e & f \\ g & h & i \\ 2g+a & 2h+b & 2i+c \end{pmatrix}$$

$$= (R_2 - \tfrac{1}{2}R_3)\det\begin{pmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{pmatrix} + R_3 - 2R_2 \det\begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}.$$

$$= 2^3 \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} - \det\begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}$$

$$= 8\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} + \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= 9$$

3. (20 points) For the following vector spaces V, determine whether or not the given set W is a subspace of V. In either case, justify your answer thoroughly.

(a) $V = \mathbb{R}^4$ and $W = \{(x_1, x_2, x_3, x_4) \in V \mid x_1 + x_2 = x_3 + x_4\}$;

(b) $V = P_4(\mathbb{R})$ (i.e., the vector space of polynomials with real coefficients, having degree at most equal to 4) and $W = \{p \in V \mid p(1)p(2) = 2p(3)\}$.

(a). W is a subspace.

① $(0,0,0,0)$ satisfies $x_1 + x_2 = 0 = x_3 + x_4$.

② Suppose $a, b \in W$. Let $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$.

Then $a+b = (a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4)$.

Now
$$\begin{aligned}(a_1+b_1)+(a_2+b_2) &= a_1+a_2+b_1+b_2 \\ &= a_3+a_4+b_3+b_4, \text{ because } a,b \in W. \\ &= (a_3+b_3)+(a_4+b_4).\end{aligned}$$

So $a+b \in W$.

③ Let $\lambda \in \mathbb{R}$ and $a = (a_1, a_2, a_3, a_4) \in W$.

Then $\lambda a = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4)$.
$$\begin{aligned}\lambda a_1 + \lambda a_2 &= \lambda(a_1+a_2) \\ &= \lambda(a_3+a_4) \quad \text{because } a \in W \\ &= \lambda a_3 + \lambda a_4.\end{aligned}$$

So $\lambda a \in W$.

Since W is non-empty, and closed under vector addition and scalar multiplication, it is a subspace.

(b) W is not a subspace.

$\{p \in V \mid p(1)p(2) = 2p(3)\}$.

Note that $p(x) = 2$, the constant function, satisfies $p(1)p(2) = (2)(2) = 2p(3)$.

But $(3p)(x)$ satisfies $(6)(6) = 36$,

whereas $2(3p(x)) = 12$.

So W is not closed under scalar multiplication.

4. (10 points) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Determine whether $A \in \operatorname{span}\{A_1, A_2, A_3\}$ or not. If so, express A as a linear combination of A_1, A_2, and A_3. Otherwise, explain why this is not possible.

Suppose $a\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + c\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We get a matrix equation:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow[R_4 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow[R_4 + R_2]{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right] \xrightarrow{R_4 + 2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

This system is inconsistent. So A is not in the span of $\{A_1, A_2, A_3\}$.

5. (10 points) Let $V = M_{3\times 2}(\mathbb{R})$ and consider the subspace

$$W = \{A \in V \,|\, a_{11} + a_{22} = 3a_{12} + 2a_{32}\}.$$

Find a spanning set for W. Completely justify your answer.

If $A \in W$, then $a_{11} = 3a_{12} + 2a_{32} - a_{22}$.

$$A = \begin{bmatrix} 3a_{12} + 2a_{32} - a_{22} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Setting each parameter $a_{12}, a_{21}, a_{22}, a_{31}, a_{32}$ to one while setting the others to zero in turn, we get the following spanning set:

$$\beta = \left\{ \begin{bmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

6. (10 points) Is the set of polynomials

$$\{x^3 - x, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6\} \subseteq P_3(\mathbb{R})$$

linearly independent? If so, justify why; if not, find an explicit linear dependence.

Suppose $a(x^3 - x) + b(2x^2 + 4) + c(-2x^3 + 3x^2 + 2x + 6) = 0$.

We get a system of equations:

$$\begin{aligned} a \quad - 2c &= 0 \\ 2b + 3c &= 0 \\ -a \quad + 2c &= 0 \\ 4b + 6c &= 0 \end{aligned}$$

write Matrix equation

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 4 & 6 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the rank $= 2 < 3 = \#$ of columns, this system has a nonzero solution. Hence the polynomials are not independent.

We get $a = 2c$ and $2b = -3c$. We can let $c = 2$, $a = 4$ and $b = -3$. Then

$$4(x^3 - x) - 3(2x^2 + 4) + 2(-2x^3 + 3x^2 + 2x + 6) = 0.$$