MTH 165

Midterm 2

03/28/2019

Name: Key

UR ID: ____________________

Circle your Instructor's Name:

Dan-Andrei Geba   Saul Lubkin   Carl McTague   Ustun Yildirim

- **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
- **The presence of notes is strictly forbidden.**
- **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work or justification is given. Please record your final numerical answers in the answer boxes, where these are provided.**
- **You are responsible for checking that this exam has all 13 pages.**

HONOR PLEDGE:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

YOUR SIGNATURE:____________________

1. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Find the matrix inverse A^{-1}.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_3]{-R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 - R_3]{R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] = [I_3 \mid A^{-1}]$$

If you have time, check your work: Does your A^{-1} satisfy $AA^{-1} = I_3$?

Answer:

$$A^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

(b) Using the answer to part (a) or otherwise, solve the system of equations

$$\begin{cases} x_1 - x_2 + 2x_3 = 1, \\ 2x_1 - 3x_2 + 3x_3 = -2, \\ x_1 - x_2 + x_3 = 4. \end{cases}$$

The vector form of this system is

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \quad \text{or} \quad A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

where A is the matrix in part (a).

So $\quad A^{-1}A\vec{x} = \vec{x} = A^{-1}\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 7 \\ -3 \end{bmatrix}$$

If you have time, check your work. Does your $\vec{x}$ satisfy $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$?

Answer: $\quad (x_1, x_2, x_3) = (14, 7, -3)$

2. (10 points) Consider the matrix

$$B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 3 \\ 0 & 2 & -1 & 0 \\ 1 & 3 & -2 & 5 \end{bmatrix}.$$

(a) Compute its determinant.

Expanding along row 3:

$$|B| = -2\begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \\ 1 & -2 & 5 \end{vmatrix} - \begin{vmatrix} 2 & -1 & 1 \\ 1 & 4 & 3 \\ 1 & 3 & 5 \end{vmatrix} = -2|B_1| - |B_2|.$$

Using the row operations $R_1 - 2R_2$ and $R_3 - R_2$ on B_1,

$$|B_1| = \begin{vmatrix} 0 & 7 & -5 \\ 1 & -2 & 3 \\ 0 & 0 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 7 & -5 \\ 0 & 2 \end{vmatrix} = -14$$

Using the same row operations on B_2,

$$|B_2| = \begin{vmatrix} 0 & -9 & -5 \\ 1 & 4 & 3 \\ 0 & -1 & 2 \end{vmatrix} = (-1)\begin{vmatrix} -9 & -5 \\ -1 & 2 \end{vmatrix} = (-1)(-23) = 23$$

$$|B| = -2(-14) - 23 = 28 - 23 = 5$$

It is not necessary to use this exact technique.

Answer:	$\det B =$ 5

(b) Argue that B is invertible and determine $\det(2B^T B^{-1} B^T)$, where B^T denotes its transpose.

Since $\det(B) = 5 \neq 0$, B is invertible.

$$\det(2B^T B^{-1} B^T) = \det(2B^T)\det(B^{-1})\det(B^T)$$
$$= 2^4 \det(B^T)\det(B^{-1})\det(B^T)$$
$$= 2^4 \det(B)\cancel{\det(B^{-1})\det}(B)$$
$$= 2^4(5)$$
$$= 80$$

Answer: $\det(2B^T B^{-1} B^T) =$ 80

3. (10 points) Let A be a 5×5 matrix satisfying $A^T = -A$. Using properties of the determinants, show that A is not invertible.

$\det(A^T) = \det(-A)$

$\det(A) = (-1)^5 \det(A)$

So $\det(A) = -\det(A)$.

Then $\det(A) = 0$. Hence A is not invertible.

4. (10 points) For the following vector spaces V, determine whether or not the given set W is a subspace of V. In either case, justify your answer thoroughly.

(a) $V = \mathbb{R}^2$ and $W = \{(x, y) \in V \,|\, x^3 \leq y^2 + 1\}$.

Let $v = (1,1)$. Since $1^3 \leq 1^2 + 1$, $v \in W$.

However, $5 \in \mathbb{R}$, and $5v = (5,5) \notin W$,

because $5^3 = 125 > 26 = 5^2 + 1$.

So W is not closed under scalar multiplication, and can not be a subspace.

Circle one answer. W is a subspace: **YES** or **NO.**

(b) V is the vector space of all real-valued functions defined on $[0,1]$ and W is the subset of V consisting of those functions f satisfying $f(\frac{1}{2}) = 2f(\frac{1}{3})$.

(1) let f_0 denote the zero vector in V.

Since $f_0(\frac{1}{2}) = 0 = 2(0) = 2f_0(\frac{1}{3})$, $f_0 \in W$.

(2.) let $f, g \in W$.

Then $(f+g)(\frac{1}{2}) = f(\frac{1}{2}) + g(\frac{1}{2})$

$= 2f(\frac{1}{3}) + 2g(\frac{1}{3})$

$= 2\left(f(\frac{1}{3}) + g(\frac{1}{3})\right)$

$= 2(f+g)(\frac{1}{3})$.

Hence W is closed under vector addition.

3.) let $\lambda \in \mathbb{R}$ and $f \in W$.

Then $(\lambda f)(\frac{1}{2}) = \lambda(f(\frac{1}{2}))$

$= \lambda\left(2f(\frac{1}{3})\right)$

$= 2\left(\lambda f(\frac{1}{3})\right)$

$= 2(\lambda f)(\frac{1}{3})$.

Hence W is closed under scalar multiplication.

Circle one answer. W is a subspace: **YES** or **NO**.

(c) $V = P_2(\mathbb{R})$ (i.e., the vector space of polynomials with real coefficients, having degree at most equal to 2) and $W = \{p \in V \mid p(1) \text{ is an integer}\}$.

Let $p(x) = x^2+1$. Then $p(1) = 2 \in \mathbb{Z}$, so $p \in W$.

However $\left(\frac{1}{3}p\right)(1) = \frac{1}{3}(2) = \frac{2}{3} \notin \mathbb{Z}$. Since $\frac{1}{3} \in \mathbb{R}$,

W is not closed under scalar multiplication,

so it is not a subspace.

Circle one answer. W is a subspace: **YES** or **NO**.

5. (10 points) Determine a spanning set S for the null space of the matrix

$$C = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 3 & 10 & -4 & 6 \\ 2 & 5 & -6 & -1 \end{bmatrix}.$$

Putting C in reduced row-echelon form:

$$\xrightarrow[R_3-2R_1]{R_2-3R_1} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{bmatrix} \xrightarrow[R_3+R_2]{R_1-3R_2} \begin{bmatrix} 1 & 0 & -8 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = s$, $x_4 = t$.

$x_1 = 8s + 8t$

$x_2 = -2s - 3t$

Then the solution set of $C\vec{x} = \vec{0}$ is

$$\text{Soln Set} = \{(8s+8t, -2s-3t, s, t) \mid s, t \in \mathbb{R}\}$$

$$= \{s(8,-2,1,0) + t(8,-3,0,1) \mid s, t \in \mathbb{R}\}$$

$$= \text{span}\{(8,-2,1,0), (8,-3,0,1)\}$$

Answer: $S = \{(8,-2,1,0), (8,-3,0,1)\}$

6. (10 points) Determine all values of the constant k such that the vectors $(1, 0, 1, k)$, $(-1, 0, k, 1)$, and $(2, 0, 1, 3)$ form a linearly independent set in $\mathbb{R}^4$.

Suppose $a(1,0,1,k) + b(-1,0,k,1) + c(2,0,1,3) = (0,0,0,0)$.

Then $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 1 & k & 1 \\ k & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. We wish to determine whether or not the system has a unique solution.

$\begin{bmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ has rank 3 if and only if the vectors are independent.

We can row-reduce to determine rank:

$$\xrightarrow[R_3 - kR_1]{R_2 - R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & k+1 & -1 \\ 0 & k+1 & 3-2k \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & k+1 & -1 \\ 0 & 0 & 2-2k \\ 0 & 0 & 0 \end{bmatrix}.$$

Rank < 3 if either $k = -1$ or $k = 2$.

We could also find the determinant of $\begin{bmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{bmatrix}$, as the row of zeros has no effect on rank.

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{vmatrix} = \begin{vmatrix} k & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ k & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & k \\ k & 1 \end{vmatrix} = 3k - 1 + 3 - k + 2(1-k^2)$$

$= -2k^2 + 2k + 4$. Setting this equal to zero, we get $(k-2)(k+1) = 0$. So rank < 3 if $k = 2$ or $k = -1$.

Answer: The desired values for k are $k \neq -1$, $k \neq 2$.

7. (10 points) Let W be the subspace of $M_2(\mathbb{R})$ consisting of all 2×2 matrices with trace zero. Find, with proof, a basis B for W.
(Note: For $A = (a_{ij})_{1\le i,j\le 2}$, its trace is given by $a_{11} + a_{22}$.)

Let M be an arbitrary matrix in W. Then $a_{22} = -a_{11}$, and a_{12} and a_{21} can be anything. So $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}$.

Then $M = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Then $W = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} = \text{span}(B)$.

To show that B is independent, suppose there are scalars c_1, c_2, c_3 such that

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and comparing entries, we determine that $c_1 = c_2 = c_3 = 0$.

So B is independent.

Answer: $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$