Math 165: Linear Algebra with Differential Equations

Midterm 2

April 3, 2018

NAME (please print legibly): Key

Your University ID Number:

Indicate your instructor and lecture time with a check in the appropriate box:

Kalyani Madhu	TR 2:00pm–3:15pm
Elizabeth Vidaurre	MW 4:50pm–6:05pm
Saul Lubkin	MW 2:00pm–3:15pm
Xuwen Chen	MW 10:25-11:40am

- **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
- **The presence of notes is strictly forbidden.**
- **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work or justification is given.**
- **You are responsible for checking that this exam has all 13 pages.**

Pledge of Honesty:

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature:

PLEASE WRITE ONLY ON THE FRONT SIDE OF EACH PAGE.

1. (10 points)

(a) (6 pts) Compute the determinant of the following matrix.

$$A = \begin{bmatrix} 0 & -2 & 5 & 3 \\ 0 & 0 & 0 & 4 \\ 1 & 6 & -11 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$$|A| = -\det\left(\begin{bmatrix} 1 & 6 & -11 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & -2 & 5 & 3 \\ 0 & 0 & -3 & 1 \end{bmatrix}\right) = -\left(-\det\begin{bmatrix} 1 & 6 & -11 & 0 \\ 0 & -2 & 5 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & -3 & 1 \end{bmatrix}\right)$$

$$= -\det\left(\begin{bmatrix} 1 & 6 & -11 & 0 \\ 0 & -2 & 5 & 3 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}\right)$$

$$= (-1)\left((1)(-2)(-3)(4)\right) = -24.$$

Answer: -24

(b) (4 pts) Solve the homogeneous system

$$-2x_2 + 5x_3 + 3x_4 = 0$$
$$4x_4 = 0$$
$$x_1 + 6x_2 - 11x_3 = 0$$
$$-3x_3 + x_4 = 0$$

Since the matrix of coefficients of this system has determinant $= -24 \neq 0$, the system has a unique solution. As it is homogeneous, the solution is $(0,0,0,0)$.

Answer: $(0,0,0,0)$.

2. (9 points) (Partial credit will not be awarded.)

Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } \det(A) = 2.$$

Find the determinant of the following matrices:

(a) $\left| \begin{bmatrix} a_{11} & 3a_{12} & a_{13} \\ 7a_{21} & 21a_{22} & 7a_{23} \\ a_{31} & 3a_{32} & a_{33} \end{bmatrix} \right| = 7 \begin{vmatrix} a_{11} & 3a_{12} & a_{13} \\ a_{21} & 3a_{22} & a_{23} \\ a_{31} & 3a_{32} & a_{33} \end{vmatrix} = (7)(3)|A| = 42$

Answer: 42

(b) $\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{12} & a_{13} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{vmatrix} = -2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= -4$

Answer: -4

(c) $(3A)(A^3)(A^T)(A^{-1})$

$|3A| = 3^3|A|$

$|A^3| = |A|^3$

$|A^T| = |A|$

$|A^{-1}| = \frac{1}{|A|}$

product $= 3^3|A| \cdot |A|^3 \cdot |A| \cdot \frac{1}{|A|}$

$= 3^3|A|^4$

$= 3^3 \cdot 2^4$

Answer: 432

4'
16
27
112
32
432

3. (12 points)

In each question below, determine whether or not the set S is a subspace of the vector space V. Justification is not required, and partial credit will not be awarded.

(a) $V = \mathbb{R}^3$, and $S = \{(x, y, z) \mid 2x + 3y = 0\}$

1.) $(0,0,0)$ satisfies $2x+3y = 2(0)+3(0)=0$.

2.) Suppose $\vec{v}_1, \vec{v}_2$ satisfy the condition.
let $\vec{v}_1 = (x_1, y_1, z_1)$ and $\vec{v}_2 = (x_2, y_2, z_2)$
then $\vec{v}_1 + \vec{v}_2 = (x_1+x_2, y_1+y_2, z_1+z_2)$
$2(x_1+x_2) + 3(y_1+y_2) = 2x_1 + 3y_1 + 2x_2 + 3y_2$
$= 0 + 0 = 0.$

Hence S is closed under scalar multiplication.

3.) let $\lambda \in \mathbb{R}$ and $\vec{v} \in S$. let $\vec{v} = (x, y, z)$. So $\lambda\vec{v} = (\lambda x, \lambda y, \lambda z)$.
Then $2(\lambda x) + 3(\lambda y) = \lambda(2x+3y) = \lambda(0) = 0.$
Hence S is closed under scalar multiplication.

☑ IS a subspace

☐ IS NOT a subspace

(b) $V = M_{2\times 2}(\mathbb{R})$ and $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ab = c + d \right\}$

let $M_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. Both are elets of S.

$M_1 + M_2 = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$. Since $(1)(1) \neq 0$, $M_1 + M_2 \notin S$.

So S is not closed under scalar multiplication, and is not a vector space,

☐ IS a subspace

☑ IS NOT a subspace

(c) $V = C^2(\mathbb{R})$ and S is the set of solutions to the homogeneous differential equation

$$y'' + y = 0.$$

1.) Let $y_0 = 0$ be the zero vector in $C^2(\mathbb{R})$.
the $y_0'' = 0$ also, so $y_0'' + y_0 = 0 + 0 = 0$.

2.) let $y_1, y_2 \in S$. Then $(y_1 + y_2)'' + (y_1 + y_2)$

$= y_1'' + y_2'' + y_1 + y_2$

$= (y_1'' + y_1) + (y_2'' + y_2) = 0 + 0 = 0.$

3.) Let $\lambda \in \mathbb{R}$ and $y \in S$. Then $(\lambda y)'' + \lambda y = \lambda y'' + \lambda y$

$= \lambda (y'' + y)$

$= \lambda (0) = 0.$

☑ IS a subspace

☐ IS NOT a subspace

(d) $V = M_{3\times 3}(\mathbb{R})$ and S is the linear span of the set

$$\left\{ \begin{bmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ -2 & 3 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 0 \\ 1 & 3 & 7 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix} \right\}$$

The linear span of any subset of vectors of any vector space V is a subspace of V.

☑ IS a subspace

☐ IS NOT a subspace

4. (9 points) Let S be the set of skew-symmetric 2×2 matrices with real entries. That is

$$S = \{A \in M_{2\times 2}(\mathbb{R}) \mid A^T = -A\}.$$

Determine whether or not S is a subspace of $M_{2\times 2}(\mathbb{R})$, and justify your answer by either proving S is a subspace or showing why it fails to be a subspace.

1.) the zero vector in $M_2(\mathbb{R})$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It's transpose is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. $-\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $\vec{0} \in S$.

2.) Suppose $M_1, M_2 \in S$. Then $(M_1 + M_2)^T = M_1^T + M_2^T$
$= -M_1 + -M_2$
$= -(M_1 + M_2)$.

3.) Suppose $\lambda \in \mathbb{R}$ and $M \in S$.
the $(\lambda M)^T = \lambda(M^T) = \lambda(-M) = -\lambda M = -(\lambda M)$.

This is a subspace, It is the linear span of

$B = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$.

5. (10 points) Let $V = P_2(\mathbb{R})$. Let S be the following subset of V.

$$S = \{\ x^2+2,\ 3,\ kx-1,\ kx^2\},$$

where k is an unknown constant.

(a) (5 pts) Does there exist a value for k such that S is a linearly independent subset of $P_2(\mathbb{R})$? Why or why not?

No. Since $V = P_2(\mathbb{R})$, $\dim(V) = 3$. There are four vectors in S (even if $k=0$), so they can't constitute an independent set.

(b) (5 pts) Does there exist a value for k such that S is a spanning set for $P_2(\mathbb{R})$? Why or why not?

Let ax^2+bx+c be an arbitrary vector in $P_2(\mathbb{R})$.

Set $c_1(x^2+2) + c_2(3) + c_3(kx-1) + c_4(kx^2) = (ax^2+bx+c)$,

then

$$c_1 + kc_4 = a$$
$$2c_1 + 3c_2 - c_3 = c$$
$$3c_3 = b.$$

Note: there are multiple ways to approach this problem

Solve this system:

$$A^{\#} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & k & a \\ 0 & 0 & k & 0 & b \\ 2 & 3 & -1 & 0 & c \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & k & a \\ 0 & 0 & k & 0 & b \\ 0 & 3 & -1 & -8 & c \end{array}\right]$$

If $k \neq 0$, the rank of this matrix will be 3, then rank A = rank $A^{\#}$, so the system has a solution. Hence S spans $P_2(\mathbb{R})$.

6. (10 points)

(a) (4 pts) Find a basis for the following subspace S of $\mathbb{R}^4$: (You do not need to prove this is a basis.)

$$S = \{(a, b, c, d) \in \mathbb{R}^4 \mid 3b - d = 0\}$$

$3b - d = 0$

$3b = d$

general vector: $(a, b, c, 3b)$

basis: $\{(1,0,0,0), (0,1,0,3), (0,0,1,0)\}$

Answer:

$B = \{(1,0,0,0), (0,1,0,3), (0,0,1,0)\}$

(1 pt) What is the dimension of S?

Answer: 3

(b) (4 pts) Find a basis for the following subspace S of $P_3(\mathbb{R})$: (You do not need to prove this is a basis.)

$$S = \{p(x) \mid p''(x) = 0\}$$

A vector in $P_3(\mathbb{R})$: $ax^3 + bx^2 + cx + d. = p(x).$

$p'(x) = 3ax^2 + 2bx + c$

$p''(x) = 6ax + 2b$

Hence vectors in S satisfy $a = b = 0$.

That is, $S = P_1(\mathbb{R})$.

Answer: $\{1, x\}$

(1 pt) What is the dimension of S?

Answer: 2

7. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -3 & 4 & 2 \\ -1 & -5 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -3 & 4 & 2 \\ 0 & -3 & 4 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -3 & 4 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

(a) (4pts) What is the dimension of the nullspace of A?

$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -3 & 4 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ has rank 3, as it's row-equivalent to $\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

By rank-nullity, dim (nullspace(A)) = $4 - 3 = \underline{\underline{1}}$.

(b) (4 pts) Find a basis for the column space of A.

The leading 1's in the row echelon matrix are in columns 1, 2, and 4. So

$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a basis for the column space of A.

(c) (2 pts) Consider the subset S of $\mathbb{R}^3$: $S = \{(1,0,-1),(2,-3,-5),(0,4,4),(-2,2,0)\}$. Find a linearly independent subset of S that has the same linear span as S.

Since span (S) = colspace (A), and $\underline{\underline{B}}$ in (b) is a basis for this space, it is linearly independent and spans (span(S)).

8. (10 points) Use the Wronskian to determine whether or not the set S of functions in linearly independent on the interval $(-\pi, \pi)$.

$$S = \{e^{-t}, \sin t\}$$

$$W(t) = \begin{vmatrix} e^{-t} & \sin t \\ -e^{-t} & \cos t \end{vmatrix} = e^{-t}\cos t - (-e^{-t}\sin t)$$
$$= e^{-t}(\sin t + \cos t),$$

We can choose $t = 0$,
as 0 is an elmt of $(-\pi, \pi)$.
$W(0) = e^{0}(\sin(0) + \cos(0)) = 1.$
As $1 \neq 0$, S is linearly independent
on $(-\pi, \pi)$.

QUESTION	VALUE	SCORE
1	10	
2	9	
3	12	
4	9	
5	10	
6	10	
7	10	
8	10	
TOTAL	80	