1. (20 pts) Determine whether each given set S is a subspace of the given vector space V. If so, give a proof; if not, state a property it fails to satisfy.

(a)(5 points) $V = \mathbb{R}^2$, and $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, x = y - 1 \right\}$.

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S \longrightarrow S$ not a subspace.

(Also fails ~~so~~ scaling + additive closure properties so these reasons also work.)

Circle final answer. S is a subspace: YES or NO?

(b)(5 points) $V = \mathbb{R}^3$, and $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x - y = z \right\} = \left\{ \begin{bmatrix} x \\ y \\ x-y \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$

$=$

One argument: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S \longrightarrow S$ not empty

$\begin{bmatrix} x \\ y \\ x-y \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ x'-y' \end{bmatrix} = \begin{bmatrix} x+x' \\ y+y' \\ (x+x')-(y+y') \end{bmatrix}$ ✓

$\alpha \begin{bmatrix} x \\ y \\ x-y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha x - \alpha y \end{bmatrix}$ ✓

So Has subspace properties

Different argument: $S = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$

$= \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$

So is a subspace as all spans are.

Circle final answer. S is a subspace: YES or NO?

(c)(5 points) $V = M_2(\mathbb{R})$, and $S = \left\{ A \in M_2(\mathbb{R}) \;\middle|\; A^T = -A \right\}$.

$$= \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R},\ \underset{\to\, a=0}{a=-a} \right\}$$

$$= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \;\middle|\; b \in \mathbb{R} \right\}$$

$$= \left\{ b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \;\middle|\; b \in \mathbb{R} \right\}$$

$= \text{Span}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$ so is a subspace as all span's are.

(Also could check subspace properties directly.)

Circle final answer. S is a subspace: YES or NO?

(d)(5 points) $V = P_2(\mathbb{R})$, and $S = \left\{ f \in P_2(\mathbb{R}) \;\middle|\; f'(0) = 1 \right\}$.

$f = 0$ is not in S

so S can't be a subspace.

Circle final answer. S is a subspace: YES or NO?

2. (16 pts) (a)(8 points) Compute the Wronskian determinant of the family of functions

$$\{t, 2t + 3t^2, t^2 - 1\}$$

on $\mathbb{R}$. Use it to decide if the family is linearly independent as a subset of $C^2(\mathbb{R})$ or not. Give justifications.

$$W = \begin{vmatrix} t & 2t+3t^2 & t^2-1 \\ 1 & 2+6t & 2t \\ 0 & 6 & 2 \end{vmatrix}$$

$$= t\begin{vmatrix} 2+6t & 2t \\ 6 & 2 \end{vmatrix} - 1\begin{vmatrix} 2t+3t^2 & t^2-1 \\ 6 & 2 \end{vmatrix} \quad \left(\begin{array}{c}\text{cofactor expansion} \\ \text{along Col 1}\end{array}\right)$$

$$= t\left(2(2+6t) - 6(2t)\right) - 1\left(2(2t+3t^2) - 6(t^2-1)\right)$$

$$= t\left(4 + \cancel{12t} - \cancel{12t}\right) - 1\left(4t + \cancel{6t^2} - \cancel{6t^2} + 6\right)$$

$$= 4t - 4t - 6 = -6 \neq 0$$

So LI (just need $\neq 0$ for some t in general)

Wronskian determinant:	-6
Circle answer. Are the functions linearly independent: YES or NO?	YES
Justification:	Wronskian nonzero.

(b)(8 points) Let $\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}$. Determine if $\vec{b} \in Span\{\vec{a}_1, \vec{a}_2\}$ or not. If so, write $\vec{b}$ explicitly as a linear combination of $\{\vec{a}_1, \vec{a}_2\}$; if not, explain why not.

$$\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 1 & -1 \\ -1 & 3 & 5 \end{array}\right] \end{array}$$

$$\Big\downarrow A_{12}(-1),\ A_{13}(1)$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 5 & 5 \end{array}\right]$$

$$\Big\downarrow A_{23}(5) \text{ then } M_2(-1)$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow A_{21}(-2)$$

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1 = -2,\ c_2 = 1$

$\therefore\ \vec{b} = -2\vec{a}_1 + \vec{a}_2$

Circle answer. $\vec{b} \in Span\{\vec{a}_1, \vec{a}_2\}$: **YES** or **NO?**

If answered YES, write $\vec{b}$ as linear combination here: $\vec{b} = -2\vec{a}_1 + \vec{a}_2$

3. (16 pts) Consider the following three vectors in the vector space $P_2(\mathbb{R})$ of real polynomials of degree less than or equal to two:

$$f_1 = 1, \qquad f_2 = 1 + x, \qquad f_3 = 1 + x + x^2.$$

(a)(8 points) Determine if $\{f_1, f_2, f_3\}$ spans $P_2(\mathbb{R})$.

Use $a_0 + a_1 x + a_2 x^2 \Leftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ isomorphism between $P_2(\mathbb{R})$ and $\mathbb{R}^3$: thus question is equivalent to

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ spanning $\mathbb{R}^3$.

This happens $\Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has a pivot in every row of RREF

$\overset{\text{as}}{\underset{\text{is square}}{\Longleftrightarrow}} \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \neq 0$

$\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \underset{\substack{\text{cofactor} \\ \text{1st} \\ \text{col}}}{=} 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1(1) = 1 \neq 0$

So Yes

Circle answer. $\{f_1, f_2, f_3\}$ **spans** $P_2(\mathbb{R})$: **YES or NO?**

(b)(8 points) Is the set $\{f_1, f_2, f_3\}$ linearly independent? If so, justify why; if not, find an explicit linear dependence.

$$\iff \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ LI in } \mathbb{R}^3$$

$$\iff \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ has pivot in every column of RREF}$$

$$\overset{\text{As}}{\underset{\text{square}}{\iff}} \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \neq 0$$

So answer is also <u>yes</u> by (a)

Circle answer. $\{f_1, f_2, f_3\}$ **is linearly independent: YES or NO?**

4. (16 pts) Let

$$A = \begin{array}{cccc} + & - & + & - \\ \end{array}$$
$$A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 5 & 0 & -2 & 6 \\ -3 & 1 & 0 & 1 \\ 0 & -3 & 0 & 2 \end{bmatrix}.$$

(a)(10 points) Find the determinant of A.

Cofactor on 3rd col:

$$\det A = -(-2)\begin{vmatrix} 3 & 2 & 0 \\ -3 & 1 & 1 \\ 0 & -3 & 2 \end{vmatrix} \overset{A_{23}(-2)}{=} +2\begin{vmatrix} 3 & 2 & 0 \\ -3 & 1 & 1 \\ 6 & -5 & 0 \end{vmatrix} \overset{\text{Cofactor on 3rd col}}{=} +2(-1)\begin{vmatrix} 3 & 2 \\ 6 & -5 \end{vmatrix}$$

$$= -2(-15-12) = -2(-27) = +54$$

$Det(A) =$ 54

(b)(6 points) Is A invertible? Explain why or why not.

Invertible? YES or NO
Explanation: det $\neq 0$

5. (16 pts)

Let

$$A = \begin{bmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{bmatrix}$$

This matrix has RREF given by:

$$U = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a)(6 points) Find a basis for the nullspace of A.

~~Cols of A corresponding to pivot cols of the 1st, 3rd, 5th~~

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \longrightarrow x_1 = 2x_2 - 4x_4 \\ \longrightarrow x_3 = x_4 \\ \longrightarrow x_5 = 0 \\ \end{array}$$

x_1, x_3, x_5 bound

x_2, x_4, free

$$\text{Nullspace } A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} \middle| \; x_2, x_4 \text{ free} \right\}$$

$$= \left\{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \middle| \;\; '' \right\}$$

$$= \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

Basis for nullspace:	$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

(b)(6 points) Find a basis for the column space of A.

Cols of A corresponding to pivot cols of U: 1st, 3rd, 5th

Basis for column space:	$\begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$

(c)(4 points) Find the rank and nullity of A.

Rank of A is:	3	(# pivots)
Nullity of A is:	2	(# columns without pivots)

6. (16 pts) Let $S_1 = \{A \in M_3(\mathbb{R}) \mid A + A^T = 0\}$ and $S_2 = \{A \in M_3(\mathbb{R}) \mid A = 3A^T\}$. Here $M_n(\mathbb{R})$ denotes the vector space of $n \times n$ real matrices.

(a)(8 points) Find a basis for S_1.

$$S_1 = \left\{ \begin{bmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{bmatrix} \middle| \, b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ b\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \middle| \, b,c,d \in \mathbb{R} \right\}$$

$$= \mathrm{Span}\left(\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{} \right)$$

LI as ± 1's in different locations & all other entries 0 so none can be combo of other two.

Basis for S_1: $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

(b)(4 points) Determine the dimension of S_1.

Dimension of S_1: 3

(c)(4 points) Determine the dimension of S_2.

$$\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{A} = 3\underbrace{\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}}_{A^T} \longrightarrow \begin{array}{l} a=3a,\ e=3e,\ i=3i \\ b=3d,\ d=3b,\ c=3g, \\ g=3c,\ h=3f,\ f=3h, \end{array}$$

$$\longrightarrow a=b=c=d=e=f=g=h=i=0$$

$$S_2 = \left\{ 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \text{Trivial vector space}$$

has basis $\{\ \}$ and dimension 0

Dimension of S_2: 0