Midterm 1, MTH 165, Spring 2020

1. An object whose temperature is 615°F is placed in a room whose temperature is 75°F. At 4 p.m., the temperature of the object is 135°F, and an hour later its temperature is 95°F. It is assumed that the temperature of the object $T = T(t)$ varies according to Newton's law of cooling, i.e.,

$$\frac{dT}{dt} = -k\,(T - T_m),$$

where T_m is the temperature of the surrounding medium and k is a positive constant. Under these assumptions, find the time at which the object was placed in the room.

2. Find the general solution of the differential equation

$$x\,y'' - y' \;=\; 4x^3.$$

(Hint: You may find useful letting $v = y'$ and rewriting the equation solely in terms of v.)

3. A 450-gal tank initially contains 50 gal of pure water. Brine enters the tank through two faucets: one containing 0.2 lb/gal of salt flows in at the rate of 3 gal/min, while the second one containing 0.6 lb/gal of salt flows in at the rate of 5 gal/min. The well-stirred mixture flows out of the tank at the rate of 4 gal/min. How much salt is in the tank just before the solution overflows?

4. We recall that a square matrix C is symmetric if $C = C^T$, where C^T denotes its transpose. Using properties of the transpose and the above definition, show that AA^T is a symmetric matrix if A is an arbitrary, not necessarily square matrix.

5. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{bmatrix}$$

(a) By performing elementary row operations, determine its reduced row-echelon form. Indicate which row operations are used in each step.

(b) Find the rank of matrix A and explain your answer.

6. Use Gauss-Jordan elimination to solve the system

$$\begin{cases} 3x_1 + 2x_2 + 3x_3 - 2x_4 = 1, \\ x_1 + x_2 + x_3 = 3, \\ x_1 + 2x_2 + x_3 - x_4 = 2. \end{cases}$$

$\boxed{\text{solutions}}$

1. An object whose temperature is 615°F is placed in a room whose temperature is 75°F. At 4 p.m., the temperature of the object is 135°F, and an hour later its temperature is 95°F. It is assumed that the temperature of the object $T = T(t)$ varies according to Newton's law of cooling, i.e.,

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where T_m is the temperature of the surrounding medium and k is a positive constant. Under these assumptions, find the time at which the object was placed in the room.

$$\int \frac{dT}{T-T_m} = \int -k\,dt + C$$

$$\ln|T-T_m| = -kt+C$$

$$T - T_m = Ce^{-kt}$$

$$T = T_m + Ce^{-kt}$$

$T_m = 75$

$T(4) = 135$

$T(5) = 95$

$T(t_0) = 615$

what is t_0?

$$T = 75 + ce^{-kt}$$

$$135 = 75 + ce^{-4k} \rightarrow 65 = ce^{-4k}$$

$$95 = 75 + ce^{-5k} \rightarrow 20 = ce^{-5k}$$

$$20 = 65e^{-k}$$

$$\ln\left(\frac{13}{4}\right) = \ln\left(\frac{65}{20}\right) = k$$

$$20 = ce^{-5k} = ce^{-5\ln\left(\frac{13}{4}\right)}$$

$$= c\left(\frac{4}{13}\right)^5$$

$$\rightarrow c = 20\left(\frac{13}{4}\right)^5$$

$$T(t) = 75 + c e^{-kt}$$

$$= 75 + 20\left(\frac{13}{4}\right)^5 e^{-t\ln\left(\frac{13}{4}\right)}$$

$$= 75 + 20\left(\frac{13}{4}\right)^5 \left(\frac{4}{13}\right)^t$$

$$= 75 + 20\left(\frac{13}{4}\right)^{5-t}$$

$$615 = T(t_0) = 75 + 20\left(\frac{13}{4}\right)^{5-t_0}$$

$$540 = 20\left(\frac{13}{4}\right)^{5-t_0}$$

$$27 = \left(\frac{13}{4}\right)^{5-t_0}$$

$$\ln(27) = (5 - t_0)\ln\left(\frac{13}{4}\right)$$

$$\frac{\ln(27)}{\ln(13/4)} = 5 - t_a$$

$$t_a = 5 - \frac{\ln(27)}{\ln\left(\frac{13}{4}\right)}$$

2. Find the general solution of the differential equation

$$x\,y'' - y' = 4x^3.$$

Let $z(x) = z = y'$

then $x z' - z = 4x^3$

$z' - \frac{1}{x} z = 4x^2 \quad (x \neq 0)$

$P(x) = -\frac{1}{x} \quad q(x) = 4x^2$

$I(x) = e^{\int P(x)dx}$

$= \frac{1}{x}$

$$z\frac{1}{x} = \int \frac{1}{x}(4x^2)\,dx + C$$

$$= x\left(\int 4x\,dx + C\right)$$

$$= 4x^2 + Cx$$

$$\to y' = 4x^2 + Cx$$

$$y = \frac{4}{3}x^3 + c_1\frac{x^2}{2} + c_2$$

$V(0)$ $S(0)=0$

3. A 450-gal tank initially contains $\boxed{\text{50 gal}}$ of pure water. Brine enters the tank through two faucets: one containing 0.2 lb/gal c_1 of salt flows in at the rate of 3 gal/min f_1, while the second one containing 0.6 lb/gal c_2 of salt flows in at the rate of 5 gal/min f_2. The well-stirred mixture flows out of the tank at the rate of 4 gal/min f_3. How much salt is in the tank just before the solution overflows?

$$\text{flow rate} = \frac{dV}{dt} = (f_1 + f_2 - f_3)$$

$$V(t) = (f_1 + f_2 - f_3)t + 50$$

$$= 4t + 50$$

Tank full when $t = 100$ min

$$\frac{dS}{dt} = f_1 c_1 + f_2 c_2 - f_3 \frac{S}{V}$$

$$= 0.6 + 3 - \left(\frac{4}{4t+50}\right) S$$

$$s' + \left(\frac{4}{4t+50}\right) = 3.6$$

1st order linear DE

$$I(t) = e^{\int \frac{4}{4t+50}\,dt} = 4t+50$$

$$s = \frac{1}{4t+50}\left(\int (4t+50)3.6\,dt + C\right)$$

$$= \frac{\frac{3.6}{8}(4t+50)^2}{(4t+50)} + \frac{C}{4t+50}$$

$$= 0.4(4t+50) + \frac{C}{4t+50}$$

general solution

$$= 1.6t + 20 + \frac{C}{4t+50}$$

$$S(0)=0 = 20+\frac{C}{50}$$

$$C = -1000$$

Particular solution

$$S(t) = 1.6t+20-\frac{1000}{4t+50}$$

amount of salt when tank is full

$$S(100) = 160+20-\frac{1000}{450}$$

$$= \boxed{180-\frac{20}{9} \quad 16.}$$

efore the solution overflows?

4. We recall that a square matrix C is symmetric if $C = C^T$, where C^T denotes its anspose. Using properties of the transpose and the above definition, show that AA^T is a mmetric matrix if A is an arbitrary, not necessarily square matrix.

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

$\rightarrow AA^T$ is symmetric

5. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{bmatrix}$$

(a) By performing elementary row operations, determine its reduced row-ech

Indicate which row operations are used in each step.

(b) Find the rank of matrix A and explain your answer.

$\frac{1}{4}R_2 \to R_2$

$\frac{1}{2}R_1 \to R_1$

$\frac{1}{2}R_3 \to R_3$

$$\longrightarrow \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 \\ 4 & 1 & 0 & 5 & 1 \\ 6 & 3 & 2 & 9 & 1 \end{bmatrix}$$

$R_1 \to R_2$

$R_2 \to R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 4 & 1 & 0 & 5 & 1 \\ 6 & 3 & 2 & 9 & 1 \end{bmatrix}$$

$-4R_1 + R_3 \to R_3$

$-6R_1 + R_4 \to R_4$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -3 & -4 & -3 & 1 \\ 0 & -3 & -4 & -3 & 1 \end{bmatrix}$$

$$-R_3 + R_4 \to R_4 \quad \begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -3 & -4 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} -R_2 + R_1 \to R_1 \\ 3R_2 + R_3 \to R_3 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tfrac{1}{2}R_3 \to R_3 \quad \begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ REF}$$

$$\begin{array}{l} R_1 + R_3 \to R_1 \\ -R_3 + R_2 \to R_2 \end{array} \longrightarrow \boxed{\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ RREF}}$$

Rank(A) = 3 since RREF of A

has 3 leading ones.

6. Use Gauss-Jordan elimination to solve the system

$$\begin{cases} 3x_1 + 2x_2 + 3x_3 - 2x_4 = 1, \\ x_1 + x_2 + x_3 = 3, \\ x_1 + 2x_2 + x_3 - x_4 = 2. \end{cases}$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 3 & -2 & 1 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 2 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{\substack{R_3 \to R_1 \\ R_1 \to R_3}} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ 3 & 2 & 3 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_3\to R_3]{-R_1+R_2\to R_2}\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -4 & 0 & 1 & -5 \end{array}\right]$$

$$\xrightarrow[2R_2+R_1\to R_1]{-4R_2+R_3\to R_3}\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 & -9 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{3}R_3\to R_3]{-R_2\to R_2}\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$-R_3+R_1 \to R_1$

$R_3+R_2 \to R_2$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$x_4 = 3$

$x_2 = 2$

$x_1 + x_3 = 1$

$x_3 = s$ free variable

$$S = \{(1-s, 2, s, 3) \mid s \in \mathbb{R}\}$$

$$= \{(1,2,0,3) + s(-1,0,1,0) \mid s \in \mathbb{R}\}$$