Midterm 1, MTH 165, Spring 2019

1. Find the equation of the curve that passes through the point $(0, 1/2)$ and whose slope at each point (x, y) is $-\frac{x}{4y}$.

2. Find the general solution of the differential equation

$$1 - y\sin x - \cos x\,\frac{dy}{dx} = 0.$$

3. A tank initially contains 10 L of a salt solution. Water flows into the tank at a rate of 3 L/min, and the well-stirred mixture flows out at a rate of 2 L/min. After 5 minutes, the concentration of salt in the tank is 0.2 g/L. Find:

(a) The amount of salt in the tank initially;

(b) The volume of the solution in the tank when the concentration of salt is 0.025 g/L.

4. Consider the matrix

$$A = \begin{bmatrix} 3 & -2 & -1 & 17 \\ 2 & 2 & -4 & 8 \\ -1 & 4 & -3 & 1 \end{bmatrix}$$

(a) By performing elementary row operations, determine its reduced row-echelon form. Indicate which row operations are used in each step.

(b) Find the rank of matrix A and explain your answer.

5. Consider the system

$$\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 0, \\ x_1 + x_2 + x_3 - x_4 = 0, \\ 4x_1 + 2x_2 - x_3 + x_4 = 0, \\ 3x_1 - x_2 + x_3 + cx_4 = 0. \end{cases}$$

(a) Find all values of the constant c for which the system is consistent.

(b) For the case when the system is consistent, use Gaussian (or Gauss-Jordan) elimination to determine the solution set.

6. Consider the differential equation

$$y' = y^2(y - 1).$$

(a) Find all of its equilibrium solutions.

(b) Determine the regions in the xy-plane where the solutions are increasing.

(c) Determine the regions in the xy-plane where the solution curves are concave up.

1. Find the equation of the curve that passes through the point $(0, 1/2)$ and whose slope at each point (x, y) is $-\frac{x}{4y}$.

$$\frac{dy}{dx} = -\frac{x}{4y} \rightarrow \int 4y\,dy = \int -x\,dx + C$$

$$2y^2 = -\frac{1}{2}x^2 + C$$

$y(0) = 1/2$

$$\frac{1}{2} = C$$

$$2y^2 = -\frac{1}{2}x^2 + \frac{1}{2}$$

$$= \frac{1}{2}(1 - x^2)$$

$$\Rightarrow y^2 = \frac{1}{4}(1 - x^2)$$

$$y = \pm \sqrt{\frac{1}{4}(1-x^2)}$$ ← positive

$$y(0) = \frac{1}{2}$$

$$y = \sqrt{\frac{1}{4}(1-x^2)}$$

$$= \boxed{\frac{1}{2}\sqrt{(1-x^2)}}$$

$$-1 \leq x \leq 1$$

2. Find the general solution of the differential equation

$$1 - y \sin x - \cos x \frac{dy}{dx} = 0.$$

$$-\cos(x)\, y' - \sin(x)\, y = -1$$

Assumed $\cos(x) \neq 0$

$$y' + \underbrace{\tan(x)}_{p(x)}\, y = \underbrace{\sec(x)}_{q(x)}$$

$$I(x) = e^{\int p(x)\,dx} = e^{\int \tan(x)\,dx}$$

$$\int \frac{\sin(x)\,dx}{\cos(x)} = -\ln|\cos(x)| = \ln|\sec(x)|$$

$$I(x) = \sec(x)$$

$$(y\,I(x))' = I(x)\,q(x)$$

we should make sure $I(x) \neq 0$

$$y\,I(x) = \int I(x)\,q(x)\,dx + C$$

$$y = \frac{1}{I(x)}\left(\int I(x)\,q(x)\,dx + C\right)$$

we assumed $\cos(x) \neq 0 \rightarrow \sec(x) \neq 0$

$$y = \frac{1}{\sec(x)}\left(\int \sec(x)\sec(x)\,dx + C\right)$$

$$= \cos(x)\left(\int \sec^2(x)\,dx + C\right)$$

$$= \cos(x)\left(\tan(x) + C\right)$$

$$\boxed{= \sin(x) + C\cdot\cos(x)}$$

r_1 $\quad$ $v(0) = 10$ $\quad$ $c_1 = 0$

3. A tank initially contains 10 L of a salt solution. Water flows into the tank at a rate of 3 L/min, and the well-stirred mixture flows out at a rate of 2 L/min. After 5 minutes, the concentration of salt in the tank is 0.2 g/L. Find:

r_2 $\quad$ $c(5) = 0.2$

$s(5)/v(5) = 0.2$

(a) The amount of salt in the tank initially;

(b) The volume of the solution in the tank when the concentration of salt is 0.025 g/L.

$$v(0) = 10$$

$$r_1 = 3 \quad r_2 = 2 \quad c_1 = 0$$

$$= 0.2 \rightarrow s(\ )$$

$$\frac{dv}{dt} = r_1 - r_2 = 3 - 2 = 1$$

$$\rightarrow v(t) = v(0) + (r_1 - r_2)t$$

$$= 10 + t$$

$$v(5) = 15$$

$$\rightarrow s(5) = 0.2(15) = 3\ g$$

$$\frac{ds}{dt} = r_1 c_1 - \frac{r_2}{V(t)} s$$

$$\frac{ds}{dt} = 0 - \frac{r_2}{10+t} s$$

$$\frac{ds}{dt} + \underbrace{\left(\frac{2}{10+t}\right)}_{p(t)} s = \underbrace{0}_{q(t)}$$

$$I(t) = e^{\int \frac{2}{10+t} dt} = e^{2 \ln |10+t|}$$

$$= (10+t)^2$$

$$s(t) = \frac{1}{(10+t)^2}\left(\int (10+t)^2 \cdot 0 \, dt + C \right)$$

$$s(t) = \frac{1}{(10+t)^2} \quad c = \frac{c}{(10+t)^2}$$

$$s(5) = 3$$

$$3 = \frac{c}{15^2} \rightarrow c = 3(15)^2$$

$$s(t) = \frac{3(15)^2}{(10+t)^2}$$

a) $s(0) = \boxed{\frac{3(15)^2}{(10)^2}} = 6.75\,g$

$$\frac{3(15)^2}{(10)^2}\,g$$

$$c(t) = \frac{s(t)}{v(t)} = \frac{3(15)^2}{(10+t)^3}$$

$$v(t) = (10+t)$$

$$0.025 = \frac{3(15)^2}{(10+t)^3}$$

$$(10+t)^3 = 120(15)^2$$

$$t = \sqrt[3]{120(15)^2} - 10$$

$$v(t) = 10 + \sqrt[3]{120(15)^2} - 10$$

$$= \sqrt[3]{120(15)^2} \quad 9/2$$

4. Consider the matrix

$$A = \begin{bmatrix} 3 & -2 & -1 & 17 \\ 2 & 2 & -4 & 8 \\ -1 & 4 & -3 & 1 \end{bmatrix}$$

(a) By performing elementary row operations, determine its reduced row-echelon form. Indicate which row operations are used in each step.

(b) Find the rank of matrix A and explain your answer.

$$\begin{matrix} R_3 \to R_1 \\ R_1 \to R_3 \end{matrix} \longrightarrow \begin{bmatrix} -1 & 4 & -3 & 1 \\ 2 & 2 & -4 & 8 \\ 3 & -2 & -1 & 17 \end{bmatrix} \xrightarrow{-R_1 \to R_1} \begin{bmatrix} 1 & -4 & 3 & -1 \\ 2 & 2 & -4 & 8 \\ 3 & -2 & -1 & 17 \end{bmatrix}$$

$$\begin{matrix} -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{matrix} \quad \begin{bmatrix} 1 & -4 & 3 & -1 \\ 0 & 10 & -10 & 10 \\ 0 & 10 & -10 & 20 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{10}R_2 \to R_2} \begin{bmatrix} 1 & -4 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 10 & -10 & 20 \end{bmatrix}$$

$$\xrightarrow[-10R_2+R_3 \to R_3]{4R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{10}R_3 \to R_3} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REF

Rank(A) = 3

$-3R_3 + R_1 \to R_1$

$-R_3 + R_2 \to R_2$

$$\boxed{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

RREF

b) Find Rank(A). Explain

$\boxed{\text{Rank}(A) = 3 \quad \text{RREF of A has 3 leading ones.}}$

5. Consider the system

$$\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 0, \\ x_1 + x_2 + x_3 - x_4 = 0, \\ 4x_1 + 2x_2 - x_3 + x_4 = 0, \\ 3x_1 - x_2 + x_3 + cx_4 = 0. \end{cases}$$

at least 1 solution

(a) Find all values of the constant c for which the system is consistent.

(b) For the case when the system is consistent, use Gaussian (or Gauss-Jordan) elimination to determine the solution set.

· homogeneous system

→ must have (0,0,0,0) as a solution → system is consistent for every c.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 4 & 2 & -1 & 1 & 0 \\ 3 & -1 & 1 & c & 0 \end{array}\right] \xrightarrow[R_1 \to R_2]{R_2 \to R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 4 & 2 & -1 & 1 & 0 \\ 3 & -1 & 1 & c & 0 \end{array}\right]$$

$$-2R_1 + R_2 \to R_2$$
$$-4R_1 + R_3 \to R_3$$
$$-3R_1 + R_4 \to R_4$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -2 & -5 & 5 & 0 \\ 0 & -4 & -2 & 3+c & 0 \end{array}\right]$$

$$-R_2 \to R_2$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & -2 & -5 & 5 & 0 \\ 0 & -4 & -2 & 3+c & 0 \end{array}\right]$$

$$\xrightarrow{\substack{-R_2+R_1\to R_1 \\ 2R_2+R_3\to R_3 \\ 4R_2+R_4\to R_4}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 10 & c-9 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{2R_3+R_1\to R_1 \\ -3R_3+R_2\to R_2 \\ -10R_3+R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & c+1 & 0 \end{array}\right]$$

$x_1 = 0$, $x_2 = 0$, $x_3 = x_4$

$(c+1)x_4 = 0$

Case 1 : $c = -1$ $\rightarrow$ $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Let $x_4 = s$ $\quad x_1 = 0$

$x_3 = x_4$

$x_2 = 0$

$c = -1$

$S = \{ (0, 0, s, s) \text{ s.t } s \in \mathbb{R} \}$

Case 2 $c \neq -1$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$c+1$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \text{RREF}$$

$$S = \{(0,0,0,0)\} \quad (c \neq -1)$$

$c = -1$

$$S = \{(0,0,s,s) \text{ s.t. } s \in \mathbb{R}\}$$

6. Consider the differential equation

$$y' = y^2(y-1).$$

(a) Find all of its equilibrium solutions.

(b) Determine the regions in the xy-plane where the solutions are increasing.

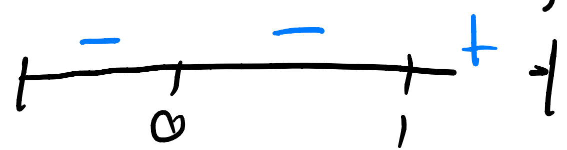
(c) Determine the regions in the xy-plane where the solution curves are concave up.

a) $y' = 0$ & $y(x) = C$ ← constant

solve $0 = y^2(y-1)$

equilibrium solutions: $y=0$, $y=1$

b) $y' > 0$, w/



increasing: $(1, \infty)$

c) $y'' > 0$

$$y' = y^2(y-1) = y^3 - y^2$$

recall that $y = y(x)$

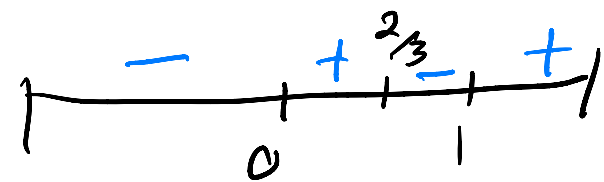
$$y'' = 3y^2 y' - 2y\,y'$$

$$= (3y^2 - 2y)\,y'$$

$$= (3y^2 - 2y)\,y^2(y-1)$$

$$= y^3(3y-2)(y-1)$$

y''



concave up: $(0, 2/3), (1, \infty)$