Math 165: ~~Calculus I~~ Linear Algebra with Differential Equations

Midterm 1

February 27, 2018

NAME (please print legibly): Solution

Your University ID Number: ________________

Indicate your instructor and lecture time with a check in the appropriate box:

Kalyani Madhu	TR 2:00pm–3:15pm
Elizabeth Vidaurre	MW 3:25pm–4:40pm
Saul Lubkin	MW 2:00pm–3:15pm
Xuwen Chen	MW 10:25-11:40am

- **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
- **The presence of notes is strictly forbidden.**
- **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work or justification is given.**
- **You are responsible for checking that this exam has all 12 pages.**

Pledge of Honesty:

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ________________

PLEASE WRITE ONLY ON THE FRONT SIDE OF EACH PAGE.

1. (10 points) Solve the initial value problem.

$$(1 - x^2)y' + xy = x \quad y(0) = 3.$$

$$(1-x^2)y' = x - xy$$
$$= x(1-y)$$
$$\frac{1}{1-y}y' = \frac{x}{1-x^2}$$

Separating,

$$\int \frac{1}{1-y}\,dy = \int \frac{x}{1-x^2}\,dx$$

LHS

let $u = 1-y$

$du = -dy$

$$-\int \frac{1}{u}\,du = -\ln|u|$$
$$= \ln|u|^{-1}$$
$$= \ln|1-y|^{-1}$$

RHS

Let $u = 1-x^2$

$du = -2x\,dx$

$-\frac{1}{2}u = x\,dx$

$$-\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln|u| + C$$
$$= \ln\frac{1}{\sqrt{|u|}} + C = \ln\frac{1}{\sqrt{|1-x^2|}} + C$$

Exponentiating,

$$e^{\ln|1-y|^{-1}} = e^{\ln\frac{1}{\sqrt{|1-x^2|}}}e^{C}$$
$$\frac{1}{|1-y|} = e^{C}\frac{1}{\sqrt{|1-x^2|}}$$
$$\sqrt{|1-x^2|} = e^{C}|1-y|$$
$$\sqrt{|1-x^2|} = c_1(1-y)$$

let $x = 0$, $y = 3$.

$$1 = -2c_1, \quad c_1 = \frac{-1}{2}$$

$$-\frac{1}{2}y = -\frac{1}{2} - \sqrt{1-x^2}$$

$$\boxed{y = 1 + 2\sqrt{1-x^2}}$$

2. (10 points) Solve the initial value problem.

$$y' - 4x = \frac{-1}{x\ln x}y, \quad y(e) = 0, \quad x > 1$$

$$y' + \frac{1}{x\ln x}y = 4x$$

Integrating factor $I = e^{\int \frac{1}{x\ln x}dx}$

let $u = \ln x$
$du = \frac{1}{x}dx$

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du = \ln|u| = \ln|\ln x| = \ln(\ln x)$$

as $x > 1$.

$$I = e^{\ln(\ln x)} = \ln x.$$

$$y\ln x = \int (\ln x)(4x)\,dx.$$

Parts: $u = \ln x$ $dv = 4x\,dx$
$du = \frac{1}{x}dx$ $v = 2x^2$

$$y\ln x = 2x^2\ln x - \int \frac{2x^2}{x}dx$$

$$y\ln x = 2x^2\ln x - x^2 + c$$

$$y = 2x^2 - \frac{x^2}{\ln x} + \frac{c}{\ln x}$$

$y(e) = 0$

$0 = 2e^2 - e^2 + c$
$= e^2 + c$

$c = -e^2$

$$\boxed{y = 2x^2 - \frac{x^2}{\ln x} - \frac{e^2}{\ln x}}$$

3. (14 points) A tank contains 100 L of pure water. A solution containing 0.1kg of salt per liter enters the tank at a rate of 5 L/min. The solution is mixed and drains from the tank at a rate of 10 L/min.

(a) (4 pts) Write down a **differential equation** whose solution gives the amount of salt in the tank at time t.

$$A'(t) = c_1 r_1 - c_2 r_2$$
$$= (.1)(5) - \frac{A(t)}{V(t)}(10)$$
$$= .5 - \frac{A(t)}{100-5t}(10)$$

$$V(t) = 100 + (c_1 - c_2)t$$
$$= 100 + (5-10)t$$
$$= 100 - 5t$$

$$\boxed{A'(t) = .5 - \frac{2A(t)}{20-t}}$$ Answer part (a)

$$A'(t) + \frac{2}{20-t}A(t) = .5$$

Integrating factor $I = e^{\int \frac{2}{20-t}dt}$

Let $u = 20-t$
$du = -dt$
$e^{-2\int \frac{1}{u}du} = e^{-2\ln(20-t)}$

$$= (20-t)^{-2}$$

$$A(t)(20-t)^{-2} = \int .5(20-t)^{-2}dt$$

$u = 20-t$
$du = -dt$

$$A(t)(20-t)^{-2} = -\frac{1}{2}\int u^{-2}du = -\frac{1}{2}u^{-1}(-1) = \frac{1}{2}(20-t)^{-1} + c.$$

This problem continues on the next page.

$$\boxed{A(t) = \frac{1}{2}(20-t) + c(20-t)^2}$$ Answer part (b)

(b) (4 pts) (Points will be awarded only if your answer is consistent with your answer in part (a).)

The **general solution** to the correct differential equation is

(1) $A(t) = \dfrac{t-20}{6} + \dfrac{c}{(20-t)^2}$, where c is an arbitrary constant.

(2) $A(t) = 10 - .5t + c(20-t)^2$, where c is an arbitrary constant.

(3) $A(t) = \frac{1}{2}(20-t) + c$, where c is an arbitrary constant.

(4) $A(t) = \dfrac{5t-20}{3} + \dfrac{c}{(100-t)^2}$, where c is an arbitrary constant.

See first page

This problem continues on the next page.

(c) (3 pts. Points will be awarded only if your answer is consistent with your answer in part (b).) For the given initial condition, the **particular solution** is

(1) $A(t) = \dfrac{5t-20}{3} + \dfrac{20(100^2)}{3(100-t)^2}$

(2) $A(t) = \dfrac{1}{2}(20-t) - 10$

(3) $A(t) = \dfrac{t-20}{6} + \dfrac{4,000}{3(20-t)^2}$

(4) $A(t) = 10 - .5t - \dfrac{(20-t)^2}{40}$

$A(t) = 10 - .5t + c(20-t)^2.$

$A(0) = 0 = 10 + c(400)$

$\boxed{C = \frac{-1}{40}}$

(d) (3 pts. Points will be awarded only if your answer is consistent with your answer in part (c).) What is the ***concentration*** of salt in the tank when $t = 10$?

(1) .05 kg/L

(2) 2.5 kg/L

(3) 7/30 kg/L

(4) If the correct answer is not given, write it below.

$C(t) = \dfrac{A(t)}{V(t)}.$ When $t = 10$: $A(10) = 10 - 5 - \dfrac{100}{40}$

$= 5 - 2.5$

$= 2.5$

$V(10) = 50$

$C(10) = \dfrac{2.5}{50} = \dfrac{25}{500} = \boxed{.05}$

4. (12 points)

(a) (5 pts.) Put matrix A in row echelon form. Show your work and label your elementary row operations.

$$A = \begin{bmatrix} 0 & 1 & -3 & k \\ -2 & 3 & k & 6 \\ 3 & -6 & -3 & -9 \end{bmatrix}$$

Divide R_3 by 3 and permute rows.

$$\begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & -3 & k \\ -2 & 3 & k & 6 \end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & -3 & k \\ 0 & -1 & k-2 & 0 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & -3 & k \\ 0 & 0 & k-5 & k \end{bmatrix}$$

(☆)

$$R_3/k-5 \quad \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & -3 & k \\ 0 & 0 & 1 & \frac{k}{k-5} \end{bmatrix}$$

(☆☆)

(b) (3 pts) If $k = 5$, what is the rank of A?

If $k=5$, we divide by k in the last step instead. if $k=5$, $k \neq 0$, so From (☆) we can see that, rank is 3.

(c) (2 pts) If $k = 0$, what is the rank of A?

In that case, looking at (☆☆), rank is 3.

5. (12 points) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(a) (3 pts) Find A^2, if it is defined.

Since A is 2×3, A^2 is not defined.

(b) (3 pts) Find A^T, if it is defined.

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

(c) (3 pts) Find AA^T, if it is defined.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}$$

(d) (3 pts) Find A^TA, if it is defined.

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

6. (12 points)

(a) Consider the following system of equations, where a and b are fixed real numbers:

$$2x_1 - 5x_2 = a$$
$$3x_1 + 4x_2 = b$$

$$\left[\begin{array}{cc|c} 2 & -5 & a \\ 3 & 4 & b \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 4 & b \\ 2 & -5 & a \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 9 & b-a \\ 2 & -5 & a \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 9 & b-a \\ 0 & -23 & -2b-2a+b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 9 & b-a \\ 0 & 1 & \frac{b+2a}{23} \end{array}\right]$$

rank $A = 2 =$ rank $A^{\#}$, regardless of a, b.

Decide whether each statement below is True of False.

1. This system has a unique solution for some values of a and b, but not for others.

   TRUE FALSE

2. This system has a unique solution regardless of a and b.

   TRUE FALSE

3. This system has no solution for some values of a and b, but has a solution for others.

   TRUE FALSE

4. This system has no solution regardless of a and b.

   TRUE FALSE

5. This system has infinitely-many solutions for some values of a and b, but not for others.

   TRUE FALSE

6. This system has infinitely-many solutions regardless of a and b.

   TRUE FALSE

(b) Consider the following system of equations, where a, b, and c are fixed real numbers:

$$x_1 - x_2 - 2x_3 = a$$
$$x_2 + x_3 = b$$
$$-x_1 + x_3 = c$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & a \\ 0 & 1 & 1 & b \\ -1 & 0 & 1 & c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -2 & a \\ 0 & 1 & 1 & b \\ 0 & -1 & -1 & c+a \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -2 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & a+b+c \end{array}\right]$$

This system is inconsistent if $a+b+c \neq 0$ and has infinitely-many solutions if $a+b+c=0$.

Decide whether each statement below is True of False.

1. This system has a unique solution for some values of a, b, and c, but not for others.

   TRUE FALSE

2. This system has a unique solution regardless of a, b, and c.

   TRUE FALSE

3. This system has no solution for some values of a, b, and c, but has a solution for others.

   TRUE FALSE

4. This system has no solution regardless of a, b, and c.

   TRUE FALSE

5. This system has infinitely-many solutions for some values of a, b, and c, but not for others.

   TRUE FALSE

6. This system has infinitely-many solutions regardless of a, b, and c.

   TRUE FALSE

7. (10 points)

(a) Find A^{-1} if

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 4 & -6 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} -1 & 2 & 0 & 1 & 0 & 0 \\ 4 & -6 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & 2 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

A^{-1}

(b) Use your answer from part (a) to find a matrix X satisfying $AX = B$ where

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

$A^{-1}AX = A^{-1}B$

$$= \begin{bmatrix} 3 & 1 & \frac{1}{2} \\ 2 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 6.5 & -1 & -\frac{1}{2} \\ 4.25 & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$