1. (18 pts) Find the solution to the initial value problem

$$2x + y^2 e^x \frac{dy}{dx} = 0, \quad y(0) = 3$$

$$y^2 e^x \frac{dy}{dx} = -2x$$

$$\int y^2 dy = \int -2x e^{-x} dx + C \qquad \left(\int u\,dv = uv - \int v\,du\right)$$

$u = x$
$du = dx$
$dv = e^{-x}dx$
$v = -e^{-x}$

$$\frac{y^3}{3} = -2\int \overset{u}{x} \overbrace{e^{-x} dx}^{dv} + C$$

$$= -2\left(-xe^{-x} + \int e^{-x} dx\right) + C$$

$$\frac{y^3}{3} = +2xe^{-x} + 2e^{-x} + C$$

$y(0)=3$: $\quad \frac{27}{3} = 0 + 2 + C \longrightarrow C = 7$

$$y = 3^{\frac{1}{3}}(2xe^{-x} + 2e^{-x} + 7)^{1/3}$$

$$= (6xe^{-x} + 6e^{-x} + 21)^{1/3}$$

Answer: $y(x) = 3^{1/3}(2xe^{-x} + 2e^{-x} + 7)^{1/3}$

$= (6xe^{-x} + 6e^{-x} + 21)^{1/3}$

2. (18 pts) Find the general solution of

$$xy' - 8y = 2x^{10}\cos(x^2)$$

for $x > 0$.

$$y' \underbrace{- \frac{8}{x}}_{= P(x)} y = \underbrace{2x^9\cos(x^2)}_{= Q(x)}$$

$$\text{Int. factor} = e^{-\int \frac{8}{x}dx} = e^{-8\ln x} = e^{\ln(x^{-8})} = x^{-8}$$

$$\therefore \frac{d}{dx}(yx^{-8}) = 2x^9\cos(x^2)x^{-8} = 2x\cos(x^2)$$

$$\therefore yx^{-8} = \int 2x\cos(x^2)\,dx + C$$

$u = x^2$

$du = 2x\,dx$

$$\int \cos(u)\,du = \sin(u)$$

$$\therefore yx^{-8} = \sin(x^2) + C$$

$$y = x^8(\sin(x^2) + C)$$

Answer: $y(x) =$ $x^8\sin(x^2) + Cx^8$

3. (12 pts) Find all values of r such that $y = t^r$ is a solution to the following differential equation for $t > 0$:

$$t^2y'' - ty' - 15y = 0$$

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

so $\quad t^r(r(r-1) - r - 15) = 0$

$\xrightarrow{t>0}$ $\quad r(r-1) - r - 15 = 0$

$r^2 - 2r - 15 = 0$

$(r-5)(r+3) = 0$

$\longrightarrow r = -3, 5$

The r values that work are: $-3, 5$

4. (12 pts) An object of mass 4 kg is projected vertically upward with an initial speed of $49m/s$. Assume the air resistance force is proportional to the square of velocity and that the magnitude of this force is 1 Newton when the speed of the object reaches $5m/s$. Using the gravitational constant $g = 9.8m/s^2$, write a differential equation for the velocity of the object and state initial conditions. (Take the positive direction to be downward.) **YOU DO NOT NEED TO SOLVE THE EQUATION.**

$$F = ma$$

$$F_{grav} + F_{airfriction} = m\frac{dv}{dt}$$

$$mg + \underbrace{\pm}_{} kv^2 = m\frac{dv}{dt}$$

Sign is + if object moving upwards ie $v<0$
and − if 〃 〃 downwards ie $v>0$
so sign is $-\text{sgn}(v)$ where $\boxed{\text{sgn}(v) = \begin{cases} +1 & \text{if } v>0 \\ -1 & \text{if } v<0 \\ 0 & \text{if } v=0 \end{cases}}$

$$g - \frac{\text{sgn}(v)kv^2}{m} = \frac{dv}{dt}$$

$kv^2 = 1$ when $v=5$
so $k = 1/25$

Final form of differential equation:

$$\frac{dv}{dt} = 9.8 - \frac{\text{sgn}(v)\,5v^2}{4}$$

Initial conditions: $v(0) = -49$

5. (20 pts)

(a) Use elementary row operations to put the matrix

$$\mathbb{A} = \begin{bmatrix} 3 & -1 & 22 \\ -1 & 5 & 2 \\ 2 & 4 & 24 \end{bmatrix}$$

into **reduced** row echelon form. **Show all steps** and indicate which row operations are being used at each step (you may do more than one operation per step).

$$\begin{bmatrix} 3 & -1 & 22 \\ -1 & 5 & 2 \\ 2 & 4 & 24 \end{bmatrix} \xrightarrow[A_{21}(3),\ A_{23}(2)]{M_2(-1)\text{ last}} \begin{bmatrix} 0 & 14 & 28 \\ 1 & -5 & -2 \\ 0 & 14 & 28 \end{bmatrix} \xrightarrow[\text{then } M_1(1/14)]{A_{13}(-1)} \begin{bmatrix} 0 & 1 & 2 \\ 1 & -5 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{A_{12}(5)} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Find the rank of the matrix $\mathbb{A}$ and explain how you know this is the rank of $\mathbb{A}$.

Rank of $\mathbb{A}$: 2

Explanation: # of pivot 1's in RREF

(c) Find the all solutions to the system $\mathbb{A}\vec{x} = \vec{0}$.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 8 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \longrightarrow \begin{array}{l} x_1, x_2 \text{ bdd}, x_3 \text{ free} \\ x_1 + 8x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -8 \\ -2 \\ 1 \end{bmatrix}$$

Solutions:

$$s \begin{bmatrix} -8 \\ -2 \\ 1 \end{bmatrix}, s \in \mathbb{R}.$$

(d) Suppose now that $\mathbb{B}$ is a 4×4 matrix with $\text{rank}(\mathbb{B}) = 4$. What can be said about the solutions to the system $\mathbb{B}\vec{x} = \vec{0}$? (i.e. how many are there and what are they?) Justify your answer.

Possible number of solutions and what they are:

$\boxed{\text{Unique solution } \vec{x} = \vec{0}}$ as RREF $\mathbb{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

So pivot in every column

& $\vec{x} = \vec{0}$ always works when homogeneous

Explanation:

Explanation

6. (20 pts) For parts (a) and (b), use Gaussian (or Gauss-Jordan) elimination to find all solutions to the given system or show that no solutions exist. Show all steps.

(a)

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -1 \\ 2x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 5x_2 - 2x_3 &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right] \xrightarrow[A_{13}(-3)]{A_{12}(-2)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{A_{31}(2),\ A_{32}(-2) \text{ then } M_3(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{A_{23}(1),\ A_{21}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -3 \end{array}\right] \xrightarrow{P_{23}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Solutions (if any): $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$

(b)

$$\begin{aligned} x_1 - 4x_2 - x_3 + x_4 &= 3 \\ 2x_1 - 8x_2 + x_3 - 4x_4 &= 9 \\ -x_1 + 4x_2 - 2x_3 + 5x_4 &= -6 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 2 & -8 & 1 & -4 & 9 \\ -1 & 4 & -2 & 5 & -6 \end{array}\right] \xrightarrow[A_{12}(-2)]{A_{13}(1)} \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & -3 & 6 & -3 \end{array}\right]$$

$$\Big\downarrow A_{23}(1) \text{ then } M_2(\tfrac{1}{3})$$

$$\left[\begin{array}{cccc|c} 1 & -4 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{A_{21}(1)} \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_3 bdd

x_2, x_4 free

$x_1 = 4x_2 + x_4 + 4$

$x_3 = 2x_4 + 1$

Solutions (if any): $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_2 + x_4 + 4 \\ x_2 \\ 2x_4 + 1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$

x_2, x_4 free