MTH 165: Linear Algebra with Differential Equations

Final Exam ANSWERS

December 21, 2014

1. (10 points) Solve the following initial value problems on $[1, \infty)$.

(a) $xyy' = x + 2$; $y(1) = 4$.

Answer:

The equation is not linear, so we must separate variables.

$$ydy = \frac{x+2}{x}dx = \left(1 + \frac{2}{x}\right) dx.$$

Integrating gives

$$\frac{y^2}{2} = \int ydy = \int \left(1 + \frac{2}{x}\right) dx = x + 2\ln(|x|) + C = x + 2\ln(x) + C.$$

Setting $x = 1$ gives

$$8 = \frac{4^2}{2} = 1 + 2 \cdot 0 + C \implies C = 7.$$

So y is the positive square root of $2x + 4\ln(x) + 14$.

Answer: $\underline{y = \sqrt{2x + 4\ln(x) + 14}}$.

(b) $\dfrac{y'}{x^3} + 4y - 1 = 0$; $y(1) = 2$.

Answer:

The equation is both linear and separable. We write it in the standard form

$$y' + 4x^3y = x^3,$$

and multiply both sides by the integrating factor

$$I = e^{\int 4x^3 dx} = e^{x^4}.$$

This gives

$$[e^{x^4}y]' = x^3e^{x^4} \implies e^{x^4}y = \int x^3e^{x^4}dx \overset{u=x^4}{=} \frac{e^{x^4}}{4} + C.$$

Setting $x = 1$ gives

$$e \cdot 2 = \frac{e}{4} + C \implies C = \frac{7e}{4}.$$

Answer: $\underline{y = \tfrac{1}{4} + \tfrac{7e}{4}e^{-x^4}}$.

2. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 2 & 5 & 0 \end{bmatrix}$.

(a) Compute the determinant of A.

Answer:

Expanding along the third column gives: $-1(1 \cdot 5 - 2 \cdot 2) = -1$.

Answer: $\det(A) = \underline{-1}$.

(b) Find A^{-1}.

Answer:

Following the standard algorithm gives

Answer: $A^{-1} = \underline{\begin{bmatrix} 5 & 0 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}}$.

(c) Use your answer to part (b) to solve the following system of equations:

$$\begin{aligned} x + 2y &= 1 \\ 2x + 4y + z &= 4 \\ 2x + 5y &= -3 \end{aligned}$$

Answer:

Applying the usual formula $A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}$ gives

$$\mathbf{x} = \begin{bmatrix} 5 & 0 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 2 \end{bmatrix}.$$

Answer: $(x, y, z) = \underline{(11, -5, 2)}$.

3. (10 points) Let $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^3$ be two row vectors, and let $k \in \mathbb{R}$ be a real number. For which values of k, if any, is the matrix

$$A = \begin{bmatrix} \mathbf{r}_2 + k\mathbf{r}_1 \\ \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{r}_1 + \mathbf{r}_2 \end{bmatrix}$$

invertible? Justify your answer.

Answer:

One of the ways to answer this question is to recall that

$$A \text{ is invertible } \iff \text{rank}(A) = 3 \iff \text{rowrank}(A) = 3.$$

The rows of A belong to the span of $\{\mathbf{r}_1, \mathbf{r}_2\}$, which has dimension at most two. Therefore

$$\text{rowrank}(A) \leq 2 < 3$$

regardless of what value k assumes.

Answer: <u>No values of k make A invertible</u>.

4. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Let $W \subseteq M_{2\times 2}(\mathbb{R})$ be the set of 2×2 real matrices that commute with A:

$$W = \{M \in M_{2\times 2}(\mathbb{R}) : AM = MA\}.$$

(a) Prove that W is a subspace.

Answer:

We verify the three standard properties.

(i) The 2×2 zero matrix 0_2 belongs to W because $A0_2 = 0_2 = 0_2 A$.

(ii) W is closed under addition: if M, N be two elements of W, then their sum $M + N$ belongs to W because

$$A(M + N) = AM + AN = MA + NA = (M + N)A.$$

(iii) W is closed under scalar multiplication: if M is an element of W and $\lambda \in \mathbb{R}$, then the scalar product (λM) belongs to W because

$$A(\lambda M) = \lambda AM = \lambda MA = (\lambda M)A.$$

(b) Find a basis for W, showing all of your work.

Answer:

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of W. Then

$$\begin{bmatrix} a+2c & b+2d \\ c+2a & d+2b \end{bmatrix} = AM = MA = \begin{bmatrix} a+2b & b+2a \\ c+2d & d+2c \end{bmatrix} \iff \begin{cases} a = d \\ b = c \end{cases}.$$

So $M = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Therefore M belongs to the span of the set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$, which is linearly independent.

Answer: A basis is $\underline{\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}}$.

5. (10 points)

(a) Suppose the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(a, b) = (a+b, 0)$.

(i) Find a matrix A such that $T\mathbf{v} = A\mathbf{v}$.

Answer:

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(ii) Find the dimension of the kernel of T.

Answer:

The vector (a, b) belongs to the nullspace of T precisely when

$$T(a, b) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff a + b = 0.$$

So a basis for the kernel is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. Therefore, its dimension is 1.

Alternatively, the rank of the matrix A in part (a) is 1, and by the rank-nullity theorem, dim ker (T) = 2 - rank (A) = 1.

(b) Suppose the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b+c \\ b+c & d \end{bmatrix}$$

(i) Find a basis for the kernel of T.

Answer:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b+c \\ b+c & d \end{bmatrix} \iff \begin{cases} a = d = 0 \\ b + c = 0 \end{cases}.$$

So a basis for the kernel is $\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$.

(ii) What is the dimension of the range of T?

Answer:

By the Rank-Nullity Theorem for linear transformations we have

$\dim(M_{2\times 2}(\mathbb{R}) = \dim(\text{range}(T)) + \dim(\text{nullity}(T)) \implies \dim(\text{range}(T)) = 4 - 1 = 3.$

(iii) Is $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ in the range of T? Briefly justify your answer.

Answer:

No because if it were $b + c$ would have to simultaneously be 2 and -2.

6. (10 points) Let

$$A = \begin{bmatrix} 10 & -12 & 8 \\ 0 & 1 & 0 \\ -8 & 12 & -6 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

Answer:

expanding along the middle row gives that the characteristic polynomial is

$p(t) = \det(A - tI) = (1-t)[(10-t)(-6-t)+64] = (1-t)(t^2-4t+4) = (1-t)(2-t)^2.$

The eigenvalues are its roots: $\lambda = 1, 2.$

(b) Find a basis for the eigenspace of each eigenvalue found in (a) and state its dimension.

Answer:

The standard algorithm gives that a basis for the $\lambda = 1$ eigenspace is $\left\{ \begin{bmatrix} 12 \\ 1 \\ -12 \end{bmatrix} \right\}$; and a basis for the $\lambda = 2$ eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

(c) Does $\mathbb{R}^3$ have a basis consisting of eigenvectors of A? Explain your answer.

Answer:

No because there are only two linearly independent eigenvectors.

7. (10 points) Find the general solution of:

$$y^{(4)} - y^{(3)} - 6y'' = 0.$$

Answer:

The auxiliary polynomial is

$$p(r) = r^4 - r^3 - 6r^2 = r^2(r^2 - r - 6) = r^2(r-3)(r+2).$$

Therefore the general solution is

$$\boxed{y = c_1 e^{-2x} + c_2 + c_3 x + c_4 e^{3x}.}$$

8. (10 points) Solve the initial value problem.

$$y'' - y = 45\cos(2x), \qquad y(0) = 13, \;\; y'(0) = 24$$

Answer:

As always $y = y_c + y_p$.

To find y_c we look at the auxiliary polynomial $p(r) = r^2 - 1 = (r+1)(r-1)$. Therefore

$$y_c = c_1 e^{-x} + c_2 e^{x}.$$

For y_p we guess $y_p = A\cos(2x) + B\sin(2x)$. Substituting in the equation gives

$$-5A\cos(2x) - 5B\sin(2x) = 45\cos(2x) \implies A = -9, B = 0.$$

So the general solution is

$$y = c_1 e^{-x} + c_2 e^{x} - 9\cos(2x).$$

The initial conditions give

$$\begin{cases} 13 &= c_1 + c_2 - 9 \\ 24 &= -c_1 + c_2 \end{cases} \implies c_1 = -1, c_2 = 23.$$

Putting everything together

$$\boxed{y = -e^{-x} + 23e^{x} - 9\cos(2x)}$$

9. (10 points) A spring with constant k of 13N/m is loaded with a mass of 9 kg and brought to equilibrium. At time 0, the mass is still in its equilibrium position but it is given a velocity of 5m/s. If the mass experiences a resistance (or damping) force in Newtons which has magnitude 12 times the speed at each point, find the position $y(t)$ of the mass at time t.

Answer:

We chose as positive the direction of the initial velocity. The position $y(t)$ satisfies the differential equation

$$my'' + cy' + k = 0 \implies 9y'' + 12y' + 13y = 0, y(0) = 0, y'(0) = 5.$$

The auxiliary polynomial is $p(r) = 9r^2 + 12r + 13$. its roots are $r = \frac{-2}{3} \pm i$. So the general solution is

$$y = c_1 e^{-2t/3}\cos(t) + c_2 e^{-2t/3}\sin(t).$$

The initial condition $y(0) = 0$ gives $c_1 = 0$. So

$$y = c_2 e^{-2t/3}\sin(t) \implies y' = c_2 e^{-2t/3}(\cos(t) - \tfrac{2}{3}\sin(t)).$$

The initial condition $y'(0) = 5$ gives $c_2 = 5$.

$$\boxed{y = 5e^{-2t/3}\sin(t)}$$

10. (10 points) Find the general solution for the first order linear system:

$$\begin{aligned} x_1' &= 2x_1 + x_2 - x_3 \\ x_2' &= x_1 - x_2 - x_3 \\ x_3' &= x_2 + x_3 \end{aligned}$$

Answer:

We must find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Expanding along the first column gives that the characteristic polynomial is

$$\det(A - tI) = (2-t)[(t^2-1)+1] - [(1-t)+1] = (2-t)(t^2-1) = (t+1)(t-1)(t-2).$$

So the eigenvalues are $\lambda = -1, 1, 2$.

Corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

So the general solution is

$$\boxed{\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}}$$