MTH 165: Linear Algebra with Differential Equations

Final Exam

December 16, 2014

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Indicate your instructor with a check in the box:

Doyle	MWF 10:25-11:15	
Friedmann	MW 14:00-15:15	
Madhu	MW 12:30-13:45	
Petridis	TR 15:25 -16:40	

- You have 3 hours to work on this exam and you are responsible for checking that this exam has all 12 pages.
- No calculators, cell phones, other electronic devices, books, or notes are allowed during this exam.
- Show all your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

By taking this exam, you are acknowledging that the following is prohibited by the College's Honesty Policy: Obtaining an examination prior to its administration. Using unauthorized aid during an examination or having such aid visible to you during an examination. Knowingly assisting someone else during an examination or not keeping your work adequately protected from copying by another.

1. (10 points) Solve the following initial value problems on $[1, \infty)$.

(a) $xyy' = x + 2$; $y(1) = 4$.

(b) $\dfrac{y'}{x^3} + 4y - 1 = 0$; $y(1) = 2$.

2. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 2 & 5 & 0 \end{bmatrix}$.

(a) Compute the determinant of A.

(b) Find A^{-1}.

(c) Use your answer to part (b) to solve the following system of equations:

$$\begin{aligned} x + 2y &= 1 \\ 2x + 4y + z &= 4 \\ 2x + 5y &= -3 \end{aligned}$$

3. (10 points) Let $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^3$ be two row vectors, and let $k \in \mathbb{R}$ be a real number. For which values of k, if any, is the matrix

$$A = \begin{bmatrix} \mathbf{r}_2 + k\mathbf{r}_1 \\ \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{r}_1 + \mathbf{r}_2 \end{bmatrix}$$

invertible? Justify your answer.

4. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Let $W \subseteq M_{2\times 2}(\mathbb{R})$ be the set of 2×2 real matrices that commute with A:

$$W = \{M \in M_{2\times 2}(\mathbb{R}) : AM = MA\}.$$

(a) Prove that W is a subspace.

(b) Find a basis for W, showing all of your work.

5. (10 points)

(a) Suppose the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(a, b) = (a+b, 0)$.

(i) Find a matrix A such that $T\mathbf{v} = A\mathbf{v}$.

(ii) Find the dimension of the kernel of T.

(b) Suppose the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b+c \\ b+c & d \end{bmatrix}$$

(i) Find a basis for the kernel of T.

(ii) What is the dimension of the range of T?

(iii) Is $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ in the range of T? Briefly justify your answer.

6. (10 points) Let

$$A = \begin{bmatrix} 10 & -12 & 8 \\ 0 & 1 & 0 \\ -8 & 12 & -6 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Find a basis for the eigenspace of each eigenvalue found in (a) and state its dimension.

(c) Does $\mathbb{R}^3$ have a basis consisting of eigenvectors of A? Explain your answer.

7. (10 points) Find the general solution of:

$$y^{(4)} - y^{(3)} - 6y'' = 0.$$

8. (10 points) Solve the initial value problem.

$$y'' - y = 45\cos(2x), \qquad y(0) = 13, \;\; y'(0) = 24$$

9. (10 points) A spring with constant k of 13N/m is loaded with a mass of 9 kg and brought to equilibrium. At time 0, the mass is still in its equilibrium position but it is given a velocity of 5m/s. If the mass experiences a resistance (or damping) force in Newtons which has magnitude 12 times the speed at each point, find the position $y(t)$ of the mass at time t.

10. (10 points) Find the general solution for the first order linear system:

$$\begin{aligned} x_1' &= 2x_1 + x_2 - x_3 \\ x_2' &= x_1 - x_2 - x_3 \\ x_3' &= x_2 + x_3 \end{aligned}$$