MTH 165

Final Exam
05/06/2019

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UR ID: ______________________

Circle your Instructor's Name:

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- **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
- **The presence of notes is strictly forbidden.**
- **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work or justification is given. Please record your final numerical answers in the answer boxes, where these are provided.**
- **Part A of the exam consists of Problems 1-7 and Part B of the exam consists of Problems 8-14.**
- **You are responsible for checking that this exam has all 28 pages.**

HONOR PLEDGE:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

YOUR SIGNATURE:______________________

Part A

1. (10 points) Find the general solution of the differential equation

$$(x^2 + 3)\,dy - (2xy - 2x)\,dx \;=\; 0.$$

Separate variables.

$$\frac{dy}{y-1} = \frac{2x}{x^2+3}\,dx$$

$$\int^y \frac{dy}{y-1} = \int^x \frac{2x}{x^2+3}\,dx \Rightarrow \ln|y-1| = \ln|x^2+3| + C$$

(For the RHS integral use the substitution $u = x^2+3$)

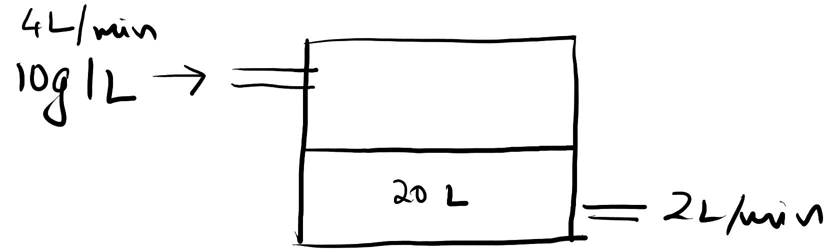
$\Rightarrow \quad |y-1| = e^C |x^2+3| \quad \Rightarrow \quad y-1 = c'(x^3+3) \qquad c' \in \mathbb{R}$

$\Rightarrow \quad y = 1 + c'(x^2+3)$ is a general solution.

(To remove the absolute value, you need to consider the cases $y-1 = e^C(x^2+3)$ and $y-1 = -e^C(x^2+3)$ and define $c' = \pm e^C$)

Answer:

2. (10 points) A tank whose volume is 40 L is initially half full with fresh water. A solution containing 10 g/L of salt is pumped into the tank at a rate of 4 L/min, and the well-stirred mixture flows out at a rate of 2 L/min. How much salt is in the tank just before the solution overflows?



$$V(t) = 20 + 2t$$

$$\frac{dA}{dt} = \left(40 - 2\frac{A(t)}{V(t)}\right) \Rightarrow \frac{dA}{dt} + \frac{2}{20+2t}A = 40$$

$$\Rightarrow \quad \frac{dA}{dt} + \frac{A}{10+t} = 40 \qquad I = e^{\int \frac{1}{10+t}dt} = e^{\log(10+t)}$$
$$= 10+t$$

$$\frac{d}{dt}\left(A(10+t)\right) = 40(10+t)$$

$$A(t)(10+t) - \overset{0}{\cancel{A(0)}}\,10 = 40\int_0^t (10+t)\,dt = 40\left[10t + \frac{t^2}{2}\right]$$

$$A(t) = \frac{400t + 20t^2}{10+t}$$

A(0)=0 follows from the fact that initially we have no salt

Answer:

NOTE: THERE IS MORE ROOM TO WORK ON THE NEXT PAGE.

Continue working on Problem 2 here.

3. (10 points) Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and assume $\det A = 1$. Find the determinants of the following matrices:

(a)

$$B = \begin{bmatrix} d-2a & e-2b & f-2c \\ 2g-3d & 2h-3e & 2i-3f \\ -4a & -4b & -4c \end{bmatrix}$$

$$\det B \overset{R_1 = R_1 - \frac{R_3}{2}}{=} \begin{vmatrix} d & e & f \\ 2g-3d & 2h-3e & 2i-3f \\ -4a & -4b & -4c \end{vmatrix} \overset{R_2 = R_2 + 3R_1}{=} \begin{vmatrix} d & e & f \\ 2g & 2h & 2i \\ -4a & -4b & -4c \end{vmatrix}$$

$$\overset{\text{move constants out}}{=} (-4)(2) \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = (-4)(-2)\underset{\text{exchange } R_1 \& R_3}{(-1)} \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$$

$$\underset{\substack{R_2 \& R_3 \\ \text{exchange}}}{=} (-8)(-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 8$$

Answer: $\det B =$

(b)

$$C = \begin{bmatrix} a+2b & d+2e & g+2h \\ 3b-5c & 3e-5f & 3h-5i \\ b & e & h \end{bmatrix}$$

$$\det(C) \overset{\swarrow}{=} \underset{R_1 = R_1 - 2R_3}{} \begin{vmatrix} a & d & g \\ 3b-5c & 3e-5f & 3h-5i \\ b & e & h \end{vmatrix}$$

$R_2 = R_2 - 3R_1$

$$\overset{\downarrow}{=} \begin{vmatrix} a & d & g \\ -5c & -5f & -5i \\ b & e & h \end{vmatrix} = (-5) \begin{vmatrix} a & d & g \\ c & f & i \\ b & e & h \end{vmatrix}$$

swap R_2 & R_3

$$\overset{\downarrow}{=} (-5)(-1) \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \underset{\uparrow}{=} 5 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$\det(A) = \det(A^T)$

$= 5$

Answer: $\det C =$

4. (10 points) For the following vector spaces V, determine whether or not the given set W is a subspace of V. In either case, justify your answer thoroughly.

(a) $V = M_3(\mathbb{R})$ (i.e., the vector space of 3×3 matrices with real elements) and

$$W = \{A \in V \mid A^T = 2A\}.$$

suppose $A \in W$ then $A^T = 2A$. If $B \in W$

$(A+B)^T = A^T + B^T = 2A + 2B = 2(A+B)$ closed under addition.

$\Rightarrow A+B \in W$

If $\lambda \in \mathbb{R}$

$(\lambda A)^T = \lambda A^T = \lambda 2A = 2(\lambda A)$

closed under scalar multiplication

YES W is a subspace

Circle one answer. W is a subspace: **YES** or **NO**.

(b) $V = C^1(\mathbb{R})$ (i.e., the vector space of differentiable functions on $\mathbb{R}$, whose derivative is continuous) and W is the subset of V consisting of those functions satisfying the differential equation

$$y' = y^2 + 2y$$

on $\mathbb{R}$.

Ugly method

(Solve the DE)

$$\frac{dy}{y^2+2y} = dx \qquad \frac{dy}{y(y+2)} = dx$$

$$\Rightarrow \frac{dy}{2}\left(\frac{1}{y} - \frac{1}{y+2}\right) = dx$$

$$\Rightarrow \frac{1}{2}\left[\log|y| - \log|y+2|\right] = x + C$$

$$\frac{y}{y+2} = c' e^{2x} \qquad c' \in \mathbb{R}$$

$$y(1 - c'e^{2x}) = 2c'e^{2x}$$

$\star 1 \longrightarrow \boxed{y = \frac{2c'e^{2x}}{1-c'e^{2x}}}$ Take $c' = 1$ and $c' = -1$

$$y_1 = \frac{2e^{2x}}{1-e^{2x}} \qquad y_2 = \frac{-2e^{2x}}{1+e^{2x}}$$

$y_1 + y_2 = \frac{2e^{2x}(1+e^{2x}-1-e^{2x})}{1-e^{4x}}$ which is not of form in ($\star 1$)

Easy: $y = -2$ is a solution BUT $(2y)(x) = -4$ is not a solution.

This shows that W is not closed under scalar multiplication.

Circle one answer. W is a subspace: **YES** or [✓ **NO**].

5. (10 points) Determine a spanning set S for the subspace of $\mathbb{R}^3$ consisting of all solutions to the linear system

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 = 0, \\ 3x_1 - 4x_2 + 7x_3 = 0. \end{cases}$$

$$A^{\#} = \begin{bmatrix} 2 & -3 & 5 & 0 \\ 3 & -4 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 7 & 0 \\ 2 & -3 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -3 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

Set $x_3 = t \Rightarrow x_2 = t$, $x_1 = -t$

So the solution set is $S = \{(-t, t, t) : t \in \mathbb{R}\}$

$= \{t(-1, 1, 1) : t \in \mathbb{R}\} = \text{span}\{(-1, 1, 1)\}$

Answer: $S = \{(-1, 1, 1)\}$

6. (10 points) Using the Wronskian, show that the functions

$$f_1(x) = \sin x, \qquad f_2(x) = \cos x, \qquad f_3(x) = \tan x,$$

are linearly independent on the interval $(-\pi/2, \pi/2)$.

$$W(x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ \cos x & -\sin x & \frac{1}{\cos^2 x} \\ -\sin x & -\cos x & -\frac{2}{\cos^3 x}\sin x \end{vmatrix}$$

$$\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$$

$$\text{Try } W(0) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 0 \quad (\text{no use})$$

$$W\left(\frac{\pi}{4}\right) = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 2 \\ -1/\sqrt{2} & -1/\sqrt{2} & -2\cdot 1\cdot 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 2 \\ -1/\sqrt{2} & -1/\sqrt{2} & -4 \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

$$\neq 0$$

So they're linearly independent.

We used $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right) \quad \tan\left(\frac{\pi}{4}\right) = 1$

7. (10 points) Find, with proof, a basis B for the subspace of $M_2(\mathbb{R})$ spanned by

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}}_{A_1}, \quad \underbrace{\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}}_{A_2}, \quad \underset{A_3}{\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}}, \quad \underset{A_4}{\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}}.$$

Corresponding equations:

$$\operatorname{span}\{A_1, A_2, A_3, A_4\} = \{aA_1 + bA_2 + cA_3 + dA_4 : a,b,c,d \in \mathbb{R}\}$$

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ 0 & -1 & 1 & 2 \\ 1 & 0 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 3 & -2 & -4 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & -11 & -13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & -11 & -13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & -13+\frac{55}{4} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & \frac{-107}{4} \end{bmatrix}$$

$\operatorname{Rank}(A) = 4 = \dim(M_2)$ So $\operatorname{span}\{A_1\ A_2\ A_3\ A_4\} = M_2$

Answer: $B = \{A_1, A_2, A_3, A_4\}$

NOTE: THERE IS MORE ROOM TO WORK ON THE NEXT PAGE.

Continue working on Problem 7 here.

Part B

8. (10 points) Let

$$A = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & 5 & 5 \end{bmatrix}.$$

(a) Determine a basis B for colspace(A).

lets find RREF (A) $A = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & 5 & 5 \end{bmatrix}$

$R_1 = R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & -2 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & -2 & 6 & 4 \end{bmatrix}$$

↓leading 1

$R_3 = R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the 1st 2 columns of A form a basis. for colspan (A)

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \right\}$$

Answer: $B =$

NOTE: THERE IS MORE ROOM TO WORK ON THE NEXT PAGE.

Continue working on part (a) here.

(b) Using the Rank-Nullity Theorem, compute the nullity of A.

rank(A) = 2　　　# of columns of A = 4

null(A) = 4 - 2 = 2

Answer: $\text{nullity}(A) =$

9. (10 points) Consider the matrix

$$C = \begin{bmatrix} -3 & 1 & 0 \\ -1 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}.$$

(a) Find all eigenvalues of C.

$$\det(\lambda I - C) = \begin{vmatrix} \lambda+3 & -1 & 0 \\ 1 & \lambda+1 & -2 \\ 0 & 0 & \lambda+2 \end{vmatrix} = (\lambda+3)\left[(\lambda+1)(\lambda+2)\right] + 1 \cdot (\lambda+2) = 0$$

$$(\lambda+2)(\lambda^2 + 4\lambda + 3 + 1) = 0 \Rightarrow (\lambda+2)(\lambda+2)(\lambda+2) = 0$$

$\Rightarrow \lambda = 2$ is the only eigenvalue. To find the eigenspace of $\lambda = -2$

we have to find the solution set of $(C - \lambda I)\vartheta = \vec{0}$

$$(C - \lambda I)^{\#} = \left[\begin{array}{ccc:c} -1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = 0 \quad x_2 = t \quad x_1 = t$

$\Rightarrow S = \{t(1, 1, 0) : t \in \mathbb{R}\}$

So the eigenspace is one dimensional

Answer:

(b) Determine a basis for each of the eigenspaces associated to C.

$B = \{ (1, 1, 0) \}$

Answer:

NOTE: THERE IS MORE ROOM TO WORK ON THE NEXT PAGE.

Continue working on part (b) here.

(c) Using the previous parts, argue whether C is defective or nondefective.

The multiplicity of -2 is 3.

$\dim(E_{-2}) = 1 < 3$

$\Rightarrow$ C is DEFECTIVE

Circle one answer. C is defective: **YES** or **NO**.

10. (10 points) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation satisfying

$$T\underbrace{(1,2,0)}_{u_1} = \underbrace{(2,-1,1)}_{z_1}, \qquad T(0,1,1) = (3,-1,-1), \qquad T(0,2,3) = (6,-5,4),$$

find the matrix A associated to T (i.e., find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.)

Write $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ Using column expansion and the equations $Au_i = z_i$ for $i = 1,2,3$ we get the equations

$$\left.\begin{aligned} v_1 + 2v_2 &= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ v_2 + v_3 &= \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \\ 2v_2 + 3v_3 &= \begin{pmatrix} 6 \\ -5 \\ 4 \end{pmatrix} \end{aligned}\right] \Rightarrow$$

$$v_3 = \begin{pmatrix} 6 \\ -5 \\ 4 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} \quad \Bigg| \quad v_2 = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \\ -7 \end{pmatrix} \qquad = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$$

$$A = \begin{bmatrix} -4 & 3 & 0 \\ -5 & 2 & -3 \\ -7 & 3 & 2 \end{bmatrix}$$

Answer:

$A =$

NOTE: THERE IS MORE ROOM TO WORK ON THE NEXT PAGE.

Continue working on Problem 10 here.

Alternatively: We know the matrix corresponding to T consists of the columns $\begin{bmatrix} Te_1 & Te_2 & Te_3 \end{bmatrix}$ Thus we only need to

write $e_1 = a_1 u_1 + b_1 u_2 + c_1 u_3$ to get $Te_1 = a\,Tu_1 + b\,Tu_2 + c\,Tu_3$

Thus $$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}}_{B^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $$\begin{bmatrix} Te_1 & Te_2 & Te_3 \end{bmatrix} = \begin{array}{c} \begin{matrix} z_1 & z_2 & z_3 \end{matrix} \\ \begin{bmatrix} 2 & 3 & 6 \\ -1 & -1 & -5 \\ 1 & -1 & 4 \end{bmatrix} \end{array} B^{-1}$$ $\det(B) = 1$

$$\operatorname{cof}(A) = \begin{bmatrix} 1 & -6 & 2 \\ 0 & 3 & -1 \\ 0 & -2 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 3 & -2 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Te_1 & Te_2 & Te_3 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 0 \\ -5 & 2 & -3 \\ -7 & 3 & 2 \end{bmatrix}$$

11. (10 points) Consider the linear transformation $T : P_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by

$$T(ax^2 + bx + c) = \begin{bmatrix} a+b & b+c \\ b+c & a-c \end{bmatrix}.$$

(a) Determine a basis for Ker(T).

$Tp = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives $\underbrace{\begin{matrix} a+b & & = 0 \\ & b+c & = 0 \\ a & - c & = 0 \end{matrix}}_{A}$ $A^{\#} = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$

$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \underset{\text{free}}{0} & 0 \end{array}\right]$ Set $c = t \Rightarrow b = -t \; a = t$

Thus $S = \{t(x^2 - x + 1) : t \in \mathbb{R}\}$ is the SOLUTION set.

and $\{x^2 - x + 1\}$ is a basis for the kernel.

Answer:

(b) Determine a basis for Rng(T).

By rank-nullity we must have $\dim(\ker(T)) + \dim(\text{Rng}(T)) = \dim(\mathbb{P}_2) = 3$

$\Rightarrow \dim(\text{Rng}(T)) = 2$

Note that any Tx is of the form $\begin{bmatrix} a+b & b+c \\ b+c & a-c \end{bmatrix}$. Set $\lambda = a+b$, $\mu = a-c$

$$= \begin{bmatrix} \lambda & \lambda-\mu \\ \lambda-\mu & \mu \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$$

Thus the basis for the range is $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \right\}$

Answer:

12. (10 points) Solve the initial value problem

$$\begin{cases} y''' + 2y'' - 4y' - 8y = 0, \\ y(0) = 0, \quad y'(0) = 6, \quad y''(0) = 8. \end{cases}$$

This is a polynomial differential operator.

$P(D) = D^3 + 2D^2 - 4D + 8$ with corresponding

polynomial $\lambda^3 + 2\lambda^2 - 4\lambda + 8 = 0$ You can guess 1 root $(\lambda = -2)$

$\Rightarrow (\lambda+2)(\lambda^2 - 4) = 0 \Rightarrow (\lambda+2)(\lambda+2)(\lambda-2)$ so the eigenvalues are

$\lambda = \pm 2$

-2 has multiplicity 2 so we have the 2 lin. indep. solutions

$$\{e^{-2t}, te^{-2t}\}$$

and 2 has multiplicity 1 with solution e^{2t}.

Thus the general solution is

$$y(x) = Ae^{-2t} + Bte^{-2t} + Ce^{2t}$$

Answer: $y(x) =$

NOTE: THERE IS MORE ROOM TO WORK ON THE NEXT PAGE.

Continue working on Problem 12 here.

$$y' = -2Ae^{-2t} + Be^{-2t} - 2Bte^{-2t} + 2Ce^{2t}$$

$$y'' = 4Ae^{-2t} - 2Be^{-2t} - 2Be^{-2t} + 4Bte^{-2t} + 4Ce^{2t}$$

Plugging in initial data

$A + C = 0$

$-2A + B + 2C = 6$

$4A - 4B + 4C = 8$

$A - B + C = 2 \Rightarrow B = -2$

$-2A + 2C = 8$

$-A + C = 4 \Rightarrow C = 2$

$\Rightarrow A = -2$

$$y(x) = -2e^{-2t} - 2te^{-2t} + 2e^{2t}$$

13. (10 points) Find the general solution to the differential equation

$$(D^2 + D - 2)y = 4\cos x - 2\sin x.$$

Homogenous eqn: $y'' + y' - 2y = 0$

Characteristic polynomial

$\lambda^2 + \lambda - 2 = 0$

Alternately you can find the eigenvalues of the associated linear system

Eigenvalues: $\det\begin{pmatrix} \lambda & -1 \\ -2 & \lambda+1 \end{pmatrix} = 0$

$\Rightarrow (\lambda+2)(\lambda-1) = 0$ $\lambda = 2$ and $\lambda = 1$ (with multiplicity 1 each)

so the homogenous part : $y(x) = Ae^{2x} + Be^{x}$ (general solution)

To find a particular solution, let $u(x) = p\cos x + q\sin x$

Then $u' = -p\sin x + q\cos x$, $u'' = -p\cos x - q\sin x$

$(-q-p-2q)\sin x$

$(-p+q-2p)\cos x$

$$u'' + u - 2u = \cos x\underbrace{(-p+q-2p)}_{4} + \sin x\underbrace{(-q-p-2q)}_{-2}$$

Answer: $y(x) =$

NOTE: THERE IS MORE ROOM TO WORK ON THE NEXT PAGE.

Continue working on Problem 13 here.

$$\Rightarrow \begin{matrix} (-p-3q) = -2 \\ -3p+q = 4 \end{matrix} \Rightarrow q+9q = 4+6 \Rightarrow q=1 \quad p=-1$$

. Hence,

$$u(x) = -\cos x + \sin x$$

Thus the general soln is of the form

$$y(x) = -\cos x + \sin x + Ae^{2x} + Be^{x}$$

14. (10 points) Determine the general solution to the system

$$\begin{cases} x_1'(t) = 2x_1(t) + 3x_3(t), \\ x_2'(t) = -4x_2(t), \\ x_3'(t) = -3x_1(t) + 2x_3(t). \end{cases}$$

Note that $x_2' = -4x_2$ has general solution $x_2 = A_2 e^{-4t}$

$$\begin{bmatrix} x_1' \\ x_3' \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

has char polynomial $(\lambda-2)(\lambda-2)+9=0$

$\Rightarrow \lambda = 2 \pm 3i$. This corresponds to

solutions of the form $x_1 = e^{-2t}\left(A_1 \cos(3t) + A_3 \sin(3t)\right)$

$x_3 = e^{-2t}\left(-A_1 \sin(3t) + A_3 \cos(3t)\right)$

So the general solution is

$x_1 = e^{-2t}\left(A_1 \cos(3t) + A_3 \sin 3t\right)$

$x_2 = A_2 e^{-4t}$

$x_3 = e^{-2t}\left(-A_1 \sin(3t) + A_3 \cos(3t)\right)$

Answer: $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} =$

NOTE: THERE IS MORE ROOM TO WORK ON THE NEXT PAGE.

Continue working on Problem 14 here.

SCRATCHWORK.