Math 165

Final

May 6, 2018

Name: Key

UR ID: ______________________

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PLEASE COPY THE HONOR PLEDGE AND SIGN:
(Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________

Part A

1. (18 points) Solve the initial value problems.

(a)

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x, \qquad y(0) = -4$$

$$(x^2+1)\frac{dy}{dx} = 6x - 3xy$$

$$= 3x(2-y)$$

$$\frac{1}{2-y}\frac{dy}{dx} = \frac{3x}{x^2+1}$$

$$\int \frac{1}{2-y}\,dy = \int \frac{3x}{x^2+1}\,dx$$

let $u = 2-y$

$du = -dy$

$-du = dy$

$-\int \frac{1}{u}\,du$

$= -\ln|u|$

$= -\ln|2-y|$

let $u = x^2+1$

$du = 2x\,dx$

$\frac{3}{2}du = 3x\,dx$

$\frac{3}{2}\int \frac{1}{u}\,du$

$= \frac{3}{2}\ln|u| + c_1$

$= \frac{3}{2}\ln|x^2+1| + C$

$$-\ln|2-y| = \frac{3}{2}\ln(x^2+1) + c$$

$$e^{\ln\left|\frac{1}{2-y}\right|} = e^{\frac{3}{2}\ln(x^2+1)}\,e^{c}$$

$$\left|\frac{1}{2-y}\right| = (x^2+1)^{\frac{3}{2}}e^{c}$$

$$\frac{1}{2-y} = (x^2+1)^{3/2}C_1$$

$y(0) = -4$, so $\frac{1}{6} = C_1$

$$2-y = \frac{1}{\frac{1}{6}(x^2+1)^{3/2}}$$

$$y = 2 - \frac{6}{(x^2+1)^{3/2}}$$

$$= 2 - 6(x^2+1)^{-\frac{3}{2}}$$

(b)

$$y' + xy = xe^{\frac{x^2}{2}},\ y(0) = 1$$

$$\boxed{\begin{aligned} I(x) &= e^{\int x\,dx} \\ &= e^{\frac{x^2}{2}} \end{aligned}}$$

$$Iy = \int I q$$

$$e^{\frac{x^2}{2}} y = \int e^{\frac{x^2}{2}} x e^{\frac{x^2}{2}}\, dx$$

$$= \int x e^{x^2}\, dx$$

Let $u = x^2$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$\int x e^{x^2} dx = \frac{1}{2} \int e^u\, du = \frac{e^u}{2}$$

$$= \frac{1}{2} e^{x^2} + c$$

$$y = e^{-\frac{x^2}{2}} \left[\frac{e^{x^2}}{2} + c \right] = \frac{1}{2} e^{\frac{x^2}{2}} + c e^{-\frac{x^2}{2}}$$

$y(0) = \frac{1}{2} + c = 1$, so $c = \frac{1}{2}$

$$y = \frac{1}{2}\left(e^{\frac{x^2}{2}} + e^{-\frac{x^2}{2}} \right)$$

2. (10 points)

(a) (8 pts) Solve the following system of equations.

$$\begin{aligned} 3x_1 + 7x_4 &= 1 \\ -x_1 + 2x_2 + x_3 &= 1 \\ 5x_1 + x_3 &= 1 \\ 2x_1 + x_3 - 7x_4 &= 1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & 7 & 1 \\ -1 & 2 & 1 & 0 & 1 \\ 5 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & -7 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 14 & 0 \\ -1 & 2 & 1 & 0 & 1 \\ 5 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & -7 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 14 & 0 \\ 0 & 2 & 0 & 14 & 1 \\ 0 & 0 & 6 & -5(14) & 1 \\ 0 & 0 & 3 & -5(7) & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 14 & 0 \\ 0 & 2 & 0 & 14 & 1 \\ 0 & 0 & 6 & -5(14) & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

This system is inconsistent. There are no solutions.

(b) (2 pts) What does your solution set tell you about whether or not the matrix of coefficients of this system is invertible?

It is <u>not</u> invertible.

3. (18 points) Please check True or False for the following questions.Partial credit will not be offered.

(a) Let $S = \{t^3 + 2, t^2 + 2, t + 2, tk\}$, where k is a real number.

(i) The set S is a basis for $P_3(\mathbb{R})$ if and only if $k \neq 1$.

☐ True If $k=0$, S is not a basis

☒ False

(ii) The set S is a basis for $P_3(\mathbb{R})$ for any $k \in \mathbb{R}$.

☐ True If $k=0$, S is not a basis.

☒ False

(iii) There does not exists a real number k such that S is a basis for $P_3(\mathbb{R})$

☐ True Using the coordinate system for $P_3(\mathbb{R})$ given by the standard basis $\{1, x, x^2, x^3\}$, we can

☒ False associate S to $\{(2,0,0,1), (2,0,1,0), (2,1,0,0), (0,k,0,0)\}$ in $\mathbb{R}^4$.

A matrix whose columns are these vectors:

$$M = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 0 & 0 & 1 & k \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$ $\det(M) = -2k \neq 0$ for any $k \neq 0$.

(b) Let A be a real 3×3 matrix whose rows, $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, satisfy

$$\mathbf{r}_1 = 2\mathbf{r}_2 + k\mathbf{r}_3,$$

for some real number k.

(i) The rank of A is 1 or 2.

☐ True If $\vec{r}_1 = \vec{r}_2 = \vec{r}_3 = \vec{0}$, the rank is 0.

☒ False

(ii) The dimension of the columnspace of A is less than 3.

☒ True Since the rows are linearly dependent,

☐ False the matrix can not have rank 3.

(iii) We can determine the dimension of the nullspace(A).

☐ True We do not know the rank, so we

☒ False can not determine the nullity.

(c) Let

$$S = \left\{ \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

(i) The set S is linearly independent in $M_2(\mathbb{R})$.

☒ True this follows from question 2.

☐ False

(ii) The set S spans $M_2(\mathbb{R})$.

☐ True $\dim(M_2(\mathbb{R})) = 4$, so a set of 3 vectors

☒ False can not span.

(iii) The set S is a basis for $M_2(\mathbb{R})$.

☐ True Since S does not span, it is not a basis.

☒ False

4. (6 points) Please check True or False for the following questions.Partial credit will not be offered.

(a) If M is a 3×3 matrix, then multiplying a column of M by 7 has the same effect on the determinant as multiplying a row by 21.

☐ True

☒ False

Multiplying a column or row by 7 multiplies the determinant by 7.

(b) If M is an $n \times n$ matrix that satisfies $MM^T = M^T M = I_n$, then $\det(A) = \pm 1$.

should be an M.

☒ True

☐ False

Since $\det(M) = \det(M^T)$

and $\det(MM^T) = \det(M)\det(M^T)$
$= \det(M^T M)$,

$\det(MM^T) = (\det(M))^2 = 1$.

so $\det M = \pm 1$.

(c) If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$.

☒ True

☐ False

$\det(AB) = \det(A)\det(B)$
$= \det(B)\det(A) = \det(BA)$.

5. (16 points)

(a) Let S be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of matrices whose trace is zero. Find a basis for this space and determine its dimension.

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary matrix in $M_2(\mathbb{R})$.

If $M \in S$, then $a+d=0$. So $d=-a$.

Hence an arbitrary matrix in S has the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

Setting each arbitrary constant to one and the others to zero in turn, we get a basis for S:

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

$\dim(S) = 3$.

(b) Let $S = \{p(x) \in P_3(\mathbb{R}) \mid p''(x) = p'(x) + x\}$. Is S a subspace of $P_3(\mathbb{R})$? Justify your answer.

This is not a subspace, as the zero vector does not have this property.

If $p(x) = \vec{0}$, then $p''(x) = p'(x) = p(x) = \vec{0}$.

So $p'(x) + x \neq \vec{0} = p''(x)$.

6. (12 points) (Justification is not required, and partial credit will not be awarded. Let A be a 5×17 matrix with real entries whose rank is 5.

(i.) The nullspace of A is a subspace of $\mathbb{R}^d$ for what value of d?

Answer: 17

The nullspace is a subspace of $\mathbb{R}^{17}$ of dimension 12.

(ii.) Would your answer in (i.) change if the rank of A was 4?

Answer: No

This would affect the dimension of the nullspace, but it would still be a subspace of $\mathbb{R}^{17}$.

(iii.) Suppose S is a spanning set for the nullspace of A. If S consists of 12 vectors, must S be a basis for the nullspace?

Answer: Yes

Since dim(nullspace) $= 17 - 5 = 12$, a spanning set of 12 vectors must be a basis.

(iv.) The columnspace of A is a subspace of $\mathbb{R}^d$ for what value of d?

Answer: 5

(v.) Suppose S is a spanning set for the columnspace of A. If S consists of 12 vectors, must S be a basis for the columnspace?

Answer: No

A basis for the columnspace would have exactly 5 vectors. A set of 12 vectors can not be independent.

(vi.) Would your answer in (v.) change if the rank of A was 4?

Answer: No

In this case, exactly 4 vectors would be needed, so 12 vectors are still too many.

Part B

7. (12 points) Let $V = M_2(\mathbb{R})$, the vector space of 2×2 real matrices.

Let $T : V \to V$ be a linear transformation given by, for any $A \in V$,

$$T(A) = RA$$

where $R = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) Find a basis for the range of T.

let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary matrix in $M_2(\mathbb{R})$.

The $T(M) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$. We see that

a matrix in the range has the form $\begin{bmatrix} s & t \\ s & t \end{bmatrix}$.

A basis is $B = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$.

(b) Find a basis for the kernel of T.

If M is in the kernel, then $c = -a$ and $d = -b$.

So a matrix in the kernel has the form $\begin{bmatrix} a & b \\ -a & -b \end{bmatrix}$.

A basis is $B = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$.

(c) Show that your answers for parts (a) and (b) are supported by the rank nullity theorem.

Since $\dim(\text{Kernel}) = 2 = \dim(\text{range})$, and

$\dim(M_2(\mathbb{R})) = 4$, and $2 + 2 = 4$, are answers

are supported by rank-nullity.

8. (16 points)

Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(a) Compute all eigenvalues of A

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = (3-\lambda)[(2-\lambda)^2 - 1]$$
$$= (3-\lambda)(4-4\lambda+\lambda^2-1)$$
$$= (3-\lambda)(\lambda^2-4\lambda+3)$$
$$= (3-\lambda)(\lambda-3)(\lambda-1)$$
$$= -(\lambda-1)(\lambda-3)^2$$

$\lambda_1 = 1, \quad \lambda_2 = 3.$

(b) For each eigenvalue in (a), find a basis for the corresponding eigenspace.

$\lambda_1 = 1$

$$\begin{bmatrix} 3-1 & 0 & 0 \\ 0 & 2-1 & -1 \\ 1 & -1 & 2-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

If $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an eigenvector, then $a = 0$, and $b - c = 0$. So $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a possible eigenvector.

$\lambda_2 = 3$

$$\begin{bmatrix} 3-3 & 0 & 0 \\ 0 & 2-3 & -1 \\ 1 & -1 & 2-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

So $a = 0$, and $c = -b$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is a possible eigenvector,

9. (12 points) Find the general solution for

$$y'' + y = 8te^t$$

$\boxed{\begin{array}{l} y_c: \\ p(r) = r^2+1 \\ r = \pm i \\ y_c = c_1 \cos t + c_2 \sin t \end{array}}$

$y_p = Ae^t + Bte^t$

$y_p' = Ae^t + Be^t + Bte^t$

$= e^t(A + B + Bt).$

$y_p'' = e^t(B) + e^t(A+B+Bt)$

$= e^t(A + 2B + Bt).$

$y_p'' + y_p = e^t(2A + 2B + 2Bt).$

Set $2(A+B) + 2Bt = 8t$

$2B = 8$

$B = 4$

$A + B = 0$

$A = -4$

<u>Solution:</u>

$y = c_1 \cos t + c_2 \sin t - 4e^t + 4te^t$

10. (16 points) Consider the differential equation

$$2x^2y'' + 5xy' + y = 0 \tag{1}$$

(a) Find two solutions of the form x^r to equation (1).

If $y = x^r$
$y' = rx^{r-1}$
$y'' = r(r-1)x^{r-2}$
$\Big\}$ so

$2x^2y'' + 5xy' + y$
$= 2x^2(r)(r-1)x^{r-2} + 5xrx^{r-1} + x^r$
$= 2x^r(r)(r-1) + 5x^r(r) + x^r$
$= x^r(2(r^2-r) + 5r + 1)$
$= x^r(2r+1)(r+1) = 0$

So $r = -\frac{1}{2}$, $r = -1$ are the desired exponents.

$y_1 = x^{-\frac{1}{2}}$, $y_2 = x^{-1}$

This equation can't be solved by techniques that work for equations with constant coefficients.

(b) Use the Wronskian to prove that the two solutions you found in part (a) are linearly independent.

$$\begin{vmatrix} x^{-\frac{1}{2}} & x^{-1} \\ -\frac{1}{2}x^{-\frac{3}{2}} & -x^{-2} \end{vmatrix} = -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{5}{2}} = x^{-\frac{5}{2}}\left(-1+\frac{1}{2}\right) \neq 0$$

for $x \neq 0$. So y_1, y_2 are independent.

(c) Give the general solution to equation (1).

$y = c_1 x^{-\frac{1}{2}} + c_2 x^{-1}$.

EXTRA PAGE. You may use this page if you run out of space. Be sure to label your problems on this page and also include a note on the original page telling the graders to look for your work here.

11. (12 points) A spring with constant k of 25 N/m is loaded with a mass of 4 kg and brought to equilibrium. It is then displaced 10 m downwards and released. If the mass experiences a resistance force in Newtons equal to 12 times the velocity at any point, find the position $y(t)$ of the mass at time t.

$$y'' + \frac{c}{m}y' + \frac{k}{M}y = F(t).$$

$$\left.\begin{array}{l} F(t)=0 \\ M=4 \\ k=25 \\ c=12 \end{array}\right\} \quad y'' + 3y' + \frac{25}{4}y = 0$$

$$p(r) = r^2 + 3r + \frac{25}{4}.$$

$$r = \frac{-3 \pm \sqrt{9 - 4\left(\frac{25}{4}\right)}}{2}$$

$$= -\frac{3}{2} \pm \frac{1}{2}\sqrt{-16}$$

$$= -\frac{3}{2} \pm 2i$$

General solution:

$$y = e^{-\frac{3}{2}t}\left(c_1 \cos 2t + c_2 \sin 2t\right).$$

$$y(0) = c_1 = 10 = y_0$$

$$y'(t) = -\frac{3}{2}e^{-\frac{3}{2}t}\left(10\cos 2t + c_2 \sin 2t\right) + e^{-\frac{3}{2}t}\left(-20\sin 2t + 2c_2 \cos 2t\right)$$

$$y'(0) = -\frac{3}{2}(10) + 2c_2 = V_0 = 0$$

$$c_2 = \frac{15}{2}$$

$$\boxed{y = e^{-\frac{3}{2}t}\left(10\cos 2t + \frac{15}{2}\sin 2t\right)}$$

12. (12 points) Solve the initial value problem. (You may leave your answer in vector form if you wish.)

$$x_1' = -2x_1 - 6x_2$$
$$x_2' = 3x_1 + 4x_2$$

$$x_1(0) = 1, \;\; x_2(0) = 1$$

$A = \begin{bmatrix} -2 & -6 \\ 3 & 4 \end{bmatrix}$ $\quad |A - \lambda I| = (-2-\lambda)(4-\lambda) + 18$

$= -8 - 4\lambda + 2\lambda + \lambda^2 + 18$

$= \lambda^2 - 2\lambda + 10$

$\lambda = \dfrac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$

$\lambda_1 = 1 + 3i$

$A - \lambda_1 I = \begin{bmatrix} -2-(1+3i) & -6 \\ 3 & 4-(1+3i) \end{bmatrix} = \begin{bmatrix} -3-3i & -6 \\ 3 & 3-3i \end{bmatrix} \to \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$

If $\vec{v} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, then $a_1 + (1-i)a_2 = 0$, so $\vec{v} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$

is an eigenvector.

To find real vectors $\vec{u}$ and $\vec{v}$,

$e^{\lambda t}\vec{v} = e^t(\cos 3t + i\sin 3t)\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$

$= e^t \begin{bmatrix} -\cos 3t - i\sin 3t + i\cos 3t - \sin 3t \\ \cos 3t + i\sin 3t \end{bmatrix}$

$= e^t\left(\begin{bmatrix} -\cos 3t - \sin 3t \\ \cos 3t \end{bmatrix} + i\begin{bmatrix} \cos 3t - \sin 3t \\ \sin 3t \end{bmatrix}\right).$

particular solution:

$\vec{X}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So $c_1 = 1$ and

$c_2 - c_1 = 1$

so $c_2 = 2$

The general solution:

$\vec{X} = e^t\left(c_1\begin{bmatrix} -\cos 3t - \sin 3t \\ \cos 3t \end{bmatrix} + c_2\begin{bmatrix} \cos 3t - \sin 3t \\ \sin 3t \end{bmatrix}\right)$

ans:

$X_1 = e^t(\cos 3t - 3\sin 3t)$

$X_2 = e^t(\cos 3t + 2\sin 3t)$